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XXV. *Of Cubic Equations and Infinite Series.*

By Charles Hutton, LL.D. F. R. S.

Read June 1, 1780.

THE following pages are not to be understood as intended to contain a complete treatise on cubic equations, with all the methods of solution that have been delivered by other writers; but they are chiefly employed on the improvements of some properties that were before but partially known, with the discovery of several others which to me appear to be new and of no small importance: for I have only slightly mentioned such of the generally known properties as were necessary to the introduction or investigation of the many curious consequences herein deduced from them.

Art. 1. Every equation, whose terms are expressed in simple integral powers, has as many roots as there are dimensions in the highest power. And when all the terms are brought to one side of the equation, and the

coefficient of the first term or highest power is  $+1$ , then the coefficient of the second term is equal to the sum of all the roots with contrary signs; the coefficient of the third term is equal to the sum of all the products made by multiplying every two of the roots together; the coefficient of the fourth term, to the sum of all the products arising from the multiplication of every three of the roots together; &c. and the last term, equal to the continual product of all the roots; the signs of all of them being supposed to be changed into the contrary signs before these multiplications are made. All this is evident from the generation of equations. And from these properties of the coefficients the following deductions are easily made.

2. If the signs of all the roots of an equation be changed, and another equation be generated from the same roots with the signs so changed; the terms of this last equation will have the same coefficients as the former, only the signs of all the even terms will be changed, but not those of the odd terms; for the coefficients of the second, fourth, and the other even terms, are made up of products consisting each of an odd number of factors; while those of the third, fifth, and other odd terms are composed of products having an even number of factors: and the change of the signs of all the factors

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produces a change in the sign of the continual product of an odd number of factors, but no change in the sign of that of an even number of factors. Wherefore, changing the signs of all the even terms, namely, the second, fourth, &c. produces no alteration in the roots, but only in their signs, the positive roots being changed into negative, and the negative into positive. But by changing any or all the signs of the odd terms, the equation will no longer have the same roots as before, but will have new roots of very different magnitudes from those of the former, unless the sign of the first term or highest power is changed also; but this term is always to be supposed to remain positive.

3. It also follows, that when any term is wanting in an equation, or the coefficient of any term equal to 0, the sum of the negative products in the coefficient of that term is equal to the sum of the positive products in the same. And if it be the second term which is wanting, then the equation has both negative and positive roots, and the sum of the negative roots is equal to the sum of the positive ones. But if it be the last term which is wanting, then one of the roots of the equation is equal to nothing. And hence arises a method of transforming any equation into another which shall want the second term: and to this latter state it will be proper to

transform every cubic equation before we attempt the resolution of it.

4. Let therefore  $x^3 + px = q$  be such a cubic equation wanting the second term, where  $p$  and  $q$  represent any numbers, positive or negative.

5. Now from the premises it follows, that this equation has three roots; that some are positive, and others negative; that two of them are of one affection, and are together equal to the third of a contrary affection, namely, either two negative roots, which are together equal to the other positive, or two positive roots equal to the third negative.

6. But the signs of the three roots are easily known from the sign of the quantity  $q$ ; the sign of the greatest root being the same with the sign of  $q$  when this quantity is on the right-hand side of the equation, and the other two roots of the contrary sign. For when  $q$  is on the same side of the equation with the other terms, it has been observed, that it is always equal to the continual product of all the roots with their signs changed; consequently  $q$  is equal to the product of all the roots under their own signs, when that quantity is on the other or right-hand side of the equation: but the product of the two less roots is always positive, because they are of the same affection, either both + or both -; and therefore

this product, drawn into the third or greatest root, will generate another product equal to  $q$ , and of the same affection with this root.

7. But the roots of equations of the above form are not only positive, negative, or nothing, but sometimes also imaginary. We have found that the greatest root is positive when  $q$  is positive, and negative when it is negative; as also that one root is  $= 0$  when  $q = 0$ , and in this case the other two roots must be equal to each other, with contrary signs. But to discover the cases in which the equation has imaginary roots, as well as many other properties of the equation, it will be proper to consider the generation of it as follows.

8. The roots of equations becoming imaginary in pairs, the number of imaginary roots is always even; and therefore the cubic equation has either two imaginary roots, or none at all; and consequently it has at least one real root. Let that root be represented by  $r$ , which may be either positive or negative, and may be any one of the real roots, when none of them are imaginary: then since any one of the roots is equal to the sum of the other two with their signs changed, the other two roots may be represented by  $-\frac{1}{2}r \pm$  some other quantity, since the sum of these two, with the signs changed, is  $= r$ . Now this supplemental quantity, which

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is to be connected with  $-\frac{1}{2}r$  by the signs + and - to compose the other two roots, will be a real quantity when those roots are real, but an imaginary one when they are imaginary, since the other part ( $-\frac{1}{2}r$ ) of those two roots is real by the hypothesis. Let this supplemental quantity be represented by  $e$  when it is real, or  $e\sqrt{-1}$  or  $\sqrt{-e^2}$  when it is imaginary: we shall use the quantity  $e$  in what follows for the real roots; and it is evident, that by changing  $e$  for  $e\sqrt{-1}$ , or  $e^2$  for  $-e^2$ , that is, by barely changing the sign of  $e^2$  wherever it is found, the expressions will become adapted to the imaginary roots. Hence then the three roots are represented by  $r$ , and  $-\frac{1}{2}r + e$ , and  $-\frac{1}{2}r - e$ ; and consequently the three equations, from whose continual multiplication by one another the cubic equation is to be generated, will be  $x - r = 0$ , and  $x + \frac{1}{2}r - e = 0$ , and  $x + \frac{1}{2}r + e = 0$ .

9. Let now these three equations be multiplied together, and there will be produced this general cubic equation wanting the second term, namely,  $x^3 - \frac{3}{4}r^2 x - r \cdot \frac{1}{4}r^2 - e^2 = 0$ , or  $x^3 - \frac{3}{4}r^2 x = r \cdot \frac{1}{4}r^2 + e^2$ , having three real roots; and if the sign of  $e^2$  be changed from - to +, it will then represent all the cases which have only one real and two imaginary roots: and from the bare inspection of this equation the following properties are easily drawn.

10. First,

10. First, we hence find, that when the equation has three real roots, the sign of the second term is always  $-$ ; for the coefficient of that term, or  $p$  is  $= -\frac{3}{4}r^2 - e^2$ , which is always negative when  $r$  and  $e$  are real quantities. And consequently when  $p$  is positive, the equation has two imaginary roots, since  $-p$  includes all the cases of three real roots. But it does not therefore follow, that when  $p$  is negative, the three roots are always real; and indeed there are imaginary roots not only whenever  $p$  is positive, but sometimes also when  $p$  is negative: for since  $p$  is  $= -\frac{3}{4}r^2 - e^2$  in all the cases of three real roots, it will be  $p = -\frac{3}{4}r^2 + e^2$  for all the cases of two imaginary roots; and it is evident, that  $p$  will be either positive or negative, according as  $e^2$  is greater or less than  $\frac{3}{4}r^2$ .

11. But to find the cases of  $-p$  when the roots are all real, and when not, will require some farther consideration; and in order to that it must be observed, that  $e^2$  ought to be positive and less than  $\frac{3}{4}r^2$ ; but the limit between the cases of real and imaginary roots is when  $e^2 = 0$ , or  $e = 0$ ; and then  $p$  becomes  $= -\frac{3}{4}r^2$ , and  $q = \frac{1}{4}r^3$ ; consequently then  $\sqrt[3]{\frac{1}{3}p} = \sqrt[3]{\frac{1}{4}r^2} = \frac{1}{64}r^6$ , which is  $= \sqrt[2]{\frac{1}{2}q} = \sqrt[3]{\frac{1}{8}r^3} = \frac{1}{64}r^6$ , that is, when  $e$  is  $= 0$ , then  $\sqrt[3]{\frac{1}{3}p} = \sqrt[2]{\frac{1}{2}q}$ , and consequently when  $\sqrt[3]{\frac{1}{3}p}$  is less than  $\sqrt[2]{\frac{1}{2}q}$ , the equation has two imaginary roots, otherwise none, the sign of  $p$  being  $-$ .

Thus



Thus then we easily perceive in all cases the nature of the roots as to real and imaginary; namely, partly from the sign of  $p$ , and partly from the relation of  $p$  to  $q$ : for the equation has always two imaginary roots when  $p$  is positive; it has also two imaginary roots when  $p$  is negative, and  $\sqrt[3]{p}$  less than  $\sqrt[3]{\frac{1}{2}q}$ ; in the other case the roots are all real, namely, when  $p$  is negative and  $\sqrt[3]{p}$  either equal to or greater than  $\sqrt[3]{\frac{1}{2}q}$ .

12. Moreover, when  $p$  is = 0, the equation has two imaginary roots; for this cannot happen but by  $-e^2$  becoming  $+e^2$ , in the value of  $p$ , and = to  $\frac{3}{4}r^2$ ; and then  $p = -\frac{3}{4}r^2 + e^2 = -\frac{3}{4}r^2 + \frac{3}{4}r^2 = 0$ , and  $q = r \cdot \sqrt[3]{\frac{1}{4}r^2 + e^2} = r \cdot \sqrt[3]{\frac{1}{4}r^2 + \frac{3}{4}r^2} = r \cdot r^2 = r^3$ , and consequently the above becomes barely  $x^3 = r^3$ , which therefore, besides one real general equation root  $x = r$ , has also two imaginary roots.

13. Hence also it again appears, that the greatest root is always of the same affection, in respect of positive and negative, with  $q$  on the right-hand side of the equation, they being either both positive or both negative together; and the other two roots of the contrary sign. For if  $r$  be the greatest root, then is  $\frac{1}{2}r$  greater than  $e$ , and  $\frac{1}{4}r^2$  greater than  $e^2$ , and  $\frac{1}{4}r^2 - e^2$  always positive, and consequently the product  $r \cdot \sqrt[3]{\frac{1}{4}r^2 - e^2}$ , or  $q$ , will have the same sign with  $r$ . But if  $r$  be one of the less roots, the contrary

trary of this will happen; for then  $\frac{1}{2}r$  is less than  $e$ , and consequently  $\frac{1}{4}r^2$  less than  $e^2$ , and so  $\frac{1}{4}r^2 - e^2$  a negative quantity, and therefore the product  $r \cdot \frac{1}{4}r^2 - e^2$ , or  $q$ , will have the sign contrary to that of  $r$ ; that is,  $q$  and the less roots have different signs, and consequently  $q$  and the greatest root the same sign, since the sign of the greatest root is always contrary to that of the other two roots.

14. Moreover, when  $q$  or  $r \cdot \frac{1}{4}r^2 - e^2$  is positive, then  $r$  denotes the greatest root; for then  $\frac{1}{4}r^2$  is greater than  $e^2$ , or  $\frac{1}{2}r$  greater than  $e$ , and  $r$  greater than either  $-\frac{1}{2}r + e$  or  $-\frac{1}{2}r - e$ . But when  $q$  or  $r \cdot \frac{1}{4}r^2 - e^2$  is negative, then  $r$  represents one of the other two roots in the equation; since then  $e$  is greater than  $\frac{1}{2}r$ , and  $-\frac{1}{2}r - e$  greater than  $r$ . Lastly, when  $q$  is between the positive and negative states, or  $q = 0$ , then  $r$  ought to be neither the greatest nor one of the less roots, if I may so speak, that is, two of the roots are equal, and the third root  $= 0$ , since then  $\frac{1}{4}r^2$  must be  $= e^2$ , or  $\frac{1}{2}r = e$ .

15. Hence it appears, that the sign of  $p$  determines the nature of the roots as to real and imaginary, and the sign of  $q$  determines the affection of the roots as to positive and negative. Let us illustrate these rules by a few examples.

16. The equation  $x^3 - 9x = 10$  has all its three roots real, because  $p = -9$  is negative, and  $\sqrt[3]{\frac{1}{3}p^3} = 3^3 = 27$  is

greater than  $\sqrt{\frac{1}{2}q}$ , which is  $= 5^2 = 25$ ; and the greatest of the roots is positive, because  $q = 10$  is positive; and the two less roots negative.

17. The equation  $x^3 - 9x = -10$  has the same three real roots as the former, but with the contrary signs, the sign of the greatest root being now negative, because  $q = -10$  is negative.

18. But the equation  $x^3 + 9x = \pm 10$  has only one real root and two imaginary roots, because  $p = 9$  is positive; and the sign of the real root is  $+$  or  $-$  according as the sign of  $q$  or  $10$  is  $+$  or  $-$ .

19. The equation  $x^3 + 6x = \pm 10$  has also two imaginary roots, and one real root, which is  $+$  or  $-$  as it is  $+10$  or  $-10$ , for the same reason as before.

20. The equation  $x^3 - 6x = \pm 10$  has also two imaginary roots, because  $\sqrt{\frac{1}{3}p}^3 = 2^3 = 8$  is less than  $\sqrt{\frac{1}{2}q}^3 = 5^2 = 25$ .

21. But the equation  $x^3 - 12x = \pm 16$  has all its roots real, because  $\sqrt{\frac{1}{3}p}^3 = 4^3 = 64$  is  $= \sqrt{\frac{1}{2}q}^3 = 8^2 = 64$ .

22. And the equation  $x^3 + 12x = \pm 16$  has only one real root, because  $p = +12$  is positive.

23. Let us now consider the other properties and relations of the roots arising from certain assumed relations between  $e$  and  $r$ , and from considering  $e$  either as real, imaginary, or nothing, that is  $e^2$  as positive, negative, or nothing.

24. When

24. When  $e$  is a real quantity, the general equation is  $x^3 - \frac{3}{4}r^2x = r \cdot \sqrt{\frac{1}{4}r^2 - e^2}$ , and all the roots are real.

25. When  $e$  is imaginary, the general equation is  $x^3 - \frac{3}{4}r^2x = r \cdot \sqrt{\frac{1}{4}r^2 + e^2}$ , and two of the roots are imaginary.

26. When  $e$  is between these two states, or  $= 0$ , the equation becomes  $x^3 - \frac{3}{4}r^2x = \frac{1}{4}r^3$ , and the root  $r = \sqrt[4]{\frac{4}{3}p} = \sqrt[3]{4q} = \frac{3q}{p}$ ; for in this case  $p = \frac{3}{4}r^2$ , and  $q = \frac{1}{4}r^3$ . Also the other two roots  $-\frac{1}{2}r \pm e$  are each  $= -\frac{1}{2}r$ .

27. Assume now any general relation between the root  $r$  and the supplemental part  $e$  of the other two roots, as suppose  $r^2 : e^2 :: 4 : n$ , or  $e^2 = \frac{n}{4}r^2$ , or  $e = \frac{1}{2}r\sqrt{n}$ , where  $n$  represents either nothing or any quantity whether positive or negative, that is, positive when  $e$  and all the three roots are real, or negative when  $e$  and two of the roots are imaginary. Substitute now  $\frac{n}{4}r^2$  instead of  $e^2$  in the general equation  $x^3 - \frac{3}{4}r^2x = r \cdot \sqrt{\frac{1}{4}r^2 - e^2}$ , and that equation will become  $x^3 - \frac{3+n}{4}r^2x = \frac{1-n}{4}r^3$ . Here then  $p = \frac{3+n}{4}r^2$ , and  $q = \frac{1-n}{4}r^3$ , and consequently the root  $r = \sqrt[3]{\frac{4p}{3+n}} = \sqrt[3]{\frac{4q}{1-n}} = \frac{n+3}{n-1} \cdot \frac{q}{p}$  expressed in three different ways. The other roots, the general values of which are  $-\frac{1}{2}r \pm e$ , become  $-\frac{1}{2}r \pm \sqrt{\frac{n}{4}r^2} = -\frac{1}{2}r \pm \frac{1}{2}r\sqrt{n} = -\frac{1}{2}r \times 1 \pm \sqrt{n}$ .

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28. Hence

28. Hence then in an easy and general manner we can represent any form or case of the general equation, with all the circumstances of the roots, by only taking, in these last formulæ, any particular number for  $n$ , either positive or negative, integral or fractional, &c. As if  $n = 1$ ; then the equation becomes  $x^3 - r^2 x = \frac{o}{2} r^3$ , or  $= 0$ , the value of  $e = \frac{1}{2} r$ , the root  $r = \sqrt{p} = \sqrt[3]{\frac{4q}{o}} = \frac{4q}{o p}$ , and the other two roots  $= -\frac{1}{2} r \cdot 1 \pm \sqrt{1} = -\frac{1}{2} r \cdot 2$  and  $-\frac{1}{2} r \cdot 0 = -r$  and  $0$ .

29. If  $n = -1$ , the equation will be  $x^3 - \frac{1}{2} r^2 x = \frac{1}{2} r^3$ , the value of  $e = \frac{1}{2} r \sqrt{-1}$ , the root  $r = \frac{q}{p} = \sqrt{2 p} = \sqrt[3]{2 q}$ , and the other two roots  $= -\frac{1}{2} r \cdot 1 \pm \sqrt{1}$  imaginary.

30. And thus, by taking several different values of  $n$ , positive and negative, the various corresponding circumstances and relations of the equation and roots will be exhibited as in the following table.

Forms of cases.	Values of $n$ .	Values of $a$ .	Forms of the equation.	Values of the root $r$ , viz. $r =$	Values of the two other roots, viz.
	$+n$	$\frac{1}{2}r\sqrt{+n}$	$x^3 - \frac{n+3}{4}r^2x = -\frac{n-1}{4}r^3$	$\frac{n+3}{n-1}p = \sqrt{\frac{4p}{n+3}} = \sqrt[3]{\frac{4q}{n-1}}$	$-\frac{1}{2}r \times \pm \sqrt{+n}$
1	$+12$	$\frac{1}{2}r\sqrt{+12}$	$x^3 - \frac{1}{2}r^2x = -\frac{1}{4}r^3$	$\frac{15q}{11p} = \sqrt{\frac{4p}{11}} = \sqrt[3]{\frac{4q}{11}}$	$\pm \sqrt{+12}$
2	$+11$	$\frac{1}{2}r\sqrt{+11}$	$x^3 - \frac{1}{2}r^2x = -\frac{10}{4}r^3$	$\frac{14q}{10p} = \sqrt{\frac{4p}{14}} = \sqrt[3]{\frac{4q}{10}}$	$\pm \sqrt{+11}$
3	$+10$	$\frac{1}{2}r\sqrt{+10}$	$x^3 - \frac{1}{2}r^2x = -\frac{9}{4}r^3$	$\frac{13q}{9p} = \sqrt{\frac{4p}{13}} = \sqrt[3]{\frac{4q}{9}}$	$\pm \sqrt{+10}$
4	$+9$	$\frac{1}{2}r\sqrt{+9}$	$x^3 - \frac{1}{2}r^2x = -\frac{8}{4}r^3$	$\frac{12q}{8p} = \sqrt{\frac{4p}{12}} = \sqrt[3]{\frac{4q}{8}}$	$\pm \sqrt{+9}$
5	$+8$	$\frac{1}{2}r\sqrt{+8}$	$x^3 - \frac{1}{2}r^2x = -\frac{7}{4}r^3$	$\frac{11q}{7p} = \sqrt{\frac{4p}{11}} = \sqrt[3]{\frac{4q}{7}}$	$\pm \sqrt{+8}$
6	$+7$	$\frac{1}{2}r\sqrt{+7}$	$x^3 - \frac{10}{4}r^2x = -\frac{6}{4}r^3$	$\frac{10q}{6p} = \sqrt{\frac{4p}{10}} = \sqrt[3]{\frac{4q}{6}}$	$\pm \sqrt{+7}$
7	$+6$	$\frac{1}{2}r\sqrt{+6}$	$x^3 - \frac{9}{4}r^2x = -\frac{5}{4}r^3$	$\frac{9q}{5p} = \sqrt{\frac{4p}{9}} = \sqrt[3]{\frac{4q}{5}}$	$\pm \sqrt{+6}$
8	$+5$	$\frac{1}{2}r\sqrt{+5}$	$x^3 - \frac{8}{4}r^2x = -\frac{4}{4}r^3$	$\frac{8q}{4p} = \sqrt{\frac{4p}{8}} = \sqrt[3]{\frac{4q}{4}}$	$\pm \sqrt{+5}$
9	$+4$	$\frac{1}{2}r\sqrt{+4}$	$x^3 - \frac{7}{4}r^2x = -\frac{3}{4}r^3$	$\frac{7q}{3p} = \sqrt{\frac{4p}{7}} = \sqrt[3]{\frac{4q}{3}}$	$\pm \sqrt{+4}$
10	$+3$	$\frac{1}{2}r\sqrt{+3}$	$x^3 - \frac{6}{4}r^2x = -\frac{2}{4}r^3$	$\frac{6q}{2p} = \sqrt{\frac{4p}{6}} = \sqrt[3]{\frac{4q}{2}}$	$\pm \sqrt{+3}$
11	$+2$	$\frac{1}{2}r\sqrt{+2}$	$x^3 - \frac{5}{4}r^2x = -\frac{1}{4}r^3$	$\frac{5q}{1p} = \sqrt{\frac{4p}{5}} = \sqrt[3]{\frac{4q}{1}}$	$\pm \sqrt{+2}$
12	$+1$	$\frac{1}{2}r\sqrt{+1}$	$x^3 - \frac{4}{4}r^2x = -\frac{0}{4}r^3$	$\frac{4q}{0p} = \sqrt{\frac{4p}{4}} = \sqrt[3]{\frac{4q}{0}}$	$\pm \sqrt{+1}$
13	$\pm 0$	$\frac{1}{2}r\sqrt{\pm 0}$	$x^3 - \frac{3}{4}r^2x = +\frac{1}{4}r^3$	$\frac{3q}{1p} = \sqrt{\frac{4p}{3}} = \sqrt[3]{\frac{4q}{1}}$	$\pm \sqrt{\pm 0}$
14	$-1$	$\frac{1}{2}r\sqrt{-1}$	$x^3 - \frac{2}{4}r^2x = +\frac{2}{4}r^3$	$\frac{2q}{2p} = \sqrt{\frac{4p}{2}} = \sqrt[3]{\frac{4q}{2}}$	$\pm \sqrt{-1}$
15	$-2$	$\frac{1}{2}r\sqrt{-2}$	$x^3 - \frac{1}{4}r^2x = +\frac{3}{4}r^3$	$\frac{1q}{3p} = \sqrt{\frac{4p}{1}} = \sqrt[3]{\frac{4q}{3}}$	$\pm \sqrt{-2}$
16	$-3$	$\frac{1}{2}r\sqrt{-3}$	$x^3 - \frac{0}{4}r^2x = +\frac{4}{4}r^3$	$\frac{0q}{4p} = \sqrt{\frac{4p}{0}} = \sqrt[3]{\frac{4q}{4}}$	$\pm \sqrt{-3}$
17	$-4$	$\frac{1}{2}r\sqrt{-4}$	$x^3 + \frac{1}{4}r^2x = +\frac{5}{4}r^3$	$\frac{1q}{5p} = \sqrt{\frac{4p}{1}} = \sqrt[3]{\frac{4q}{5}}$	$\pm \sqrt{-4}$
18	$-5$	$\frac{1}{2}r\sqrt{-5}$	$x^3 + \frac{2}{4}r^2x = +\frac{6}{4}r^3$	$\frac{2q}{6p} = \sqrt{\frac{4p}{2}} = \sqrt[3]{\frac{4q}{6}}$	$\pm \sqrt{-5}$
19	$-6$	$\frac{1}{2}r\sqrt{-6}$	$x^3 + \frac{3}{4}r^2x = +\frac{7}{4}r^3$	$\frac{3q}{7p} = \sqrt{\frac{4p}{3}} = \sqrt[3]{\frac{4q}{7}}$	$\pm \sqrt{-6}$
20	$-7$	$\frac{1}{2}r\sqrt{-7}$	$x^3 + \frac{4}{4}r^2x = +\frac{8}{4}r^3$	$\frac{4q}{8p} = \sqrt{\frac{4p}{4}} = \sqrt[3]{\frac{4q}{8}}$	$\pm \sqrt{-7}$
21	$-8$	$\frac{1}{2}r\sqrt{-8}$	$x^3 + \frac{5}{4}r^2x = +\frac{9}{4}r^3$	$\frac{5q}{9p} = \sqrt{\frac{4p}{5}} = \sqrt[3]{\frac{4q}{9}}$	$\pm \sqrt{-8}$
22	$-9$	$\frac{1}{2}r\sqrt{-9}$	$x^3 + \frac{6}{4}r^2x = +\frac{10}{4}r^3$	$\frac{6q}{10p} = \sqrt{\frac{4p}{6}} = \sqrt[3]{\frac{4q}{10}}$	$\pm \sqrt{-9}$
23	$-10$	$\frac{1}{2}r\sqrt{-10}$	$x^3 + \frac{7}{4}r^2x = +\frac{11}{4}r^3$	$\frac{7q}{11p} = \sqrt{\frac{4p}{7}} = \sqrt[3]{\frac{4q}{11}}$	$\pm \sqrt{-10}$
24	$-11$	$\frac{1}{2}r\sqrt{-11}$	$x^3 + \frac{8}{4}r^2x = +\frac{12}{4}r^3$	$\frac{8q}{12p} = \sqrt{\frac{4p}{8}} = \sqrt[3]{\frac{4q}{12}}$	$\pm \sqrt{-11}$
25	$-12$	$\frac{1}{2}r\sqrt{-12}$	$x^3 + \frac{9}{4}r^2x = +\frac{13}{4}r^3$	$\frac{9q}{13p} = \sqrt{\frac{4p}{9}} = \sqrt[3]{\frac{4q}{13}}$	$\pm \sqrt{-12}$
	$-n$	$\frac{1}{2}r\sqrt{-n}$	$x^3 + \frac{n-3}{4}r^2x = +\frac{n+1}{4}r^3$	$\frac{n-3}{n+1}p = \sqrt{\frac{4p}{n-3}} = \sqrt[3]{\frac{4q}{n+1}}$	$\pm \sqrt{-n}$

31. From the bare inspection of this table several useful and curious observations may be made. And first it appears, that when  $q$  is positive, as in all the forms after the 12th,  $r$  is the greatest root; but when  $q$  is negative, or in all the cases to the 12th,  $r$  is one of the less roots.

32. In all cases before the 4th form  $r$  is the least root, because  $\frac{\sqrt{10-1}}{2}$ , or  $\frac{\sqrt{11-1}}{2}$ , &c. is always greater than 1; and in all such forms  $\frac{1}{2}q^2$  is less than  $\frac{1}{3}p^3$ ; but the former approaches nearer and nearer to an equality with the latter till the 4th form, where  $\frac{1}{2}q^2$  is become  $= \frac{1}{3}p^3$ , and  $r$  is then equal to one of the other roots, because  $\frac{\sqrt{9-1}}{2} = \frac{2}{2} = 1$ .

33. From hence  $r$  becomes the middle root, and continues so to the 12th form, where it becomes equal to what has hitherto been the greatest root, and the other root becomes at this place  $= 0$ ; and  $\frac{1}{2}q^2$  has decreased from the 4th form all the way more and more in respect of  $\frac{1}{3}p^3$ , till at this 12th form it has become  $= 0$ , or infinitely less than  $\frac{1}{3}p^3$ .

34. From this place  $r$  becomes the greatest root, the sign of  $q$  changes to +, and  $\frac{1}{2}q^2$  again increases in respect of  $\frac{1}{3}p^3$ , till at the 13th case it becomes again equal to it, and the two less roots equal to each other, like as at the 4th form.

35. From hence  $\sqrt[1]{q}^2$  becomes greater than  $\sqrt[1]{p}^3$ , and increases more and more in respect of it, till at the 16th step where  $p$  is = 0, or  $\sqrt[1]{q}^2$  infinitely greater than  $\sqrt[1]{p}^3$ .

36. From this place the sign of  $p$  becomes +, and  $\sqrt[1]{q}^2$  continually decreases in respect of  $\sqrt[1]{p}^3$  to infinity.

37. By help of this table we may find the roots of any cubic equation  $x^3 \mp px = q$  whenever we can assign the relation between  $\sqrt{p}$  and  $\sqrt[3]{q}$ . For since one root  $r$  is always  $= \frac{n \pm 3 \cdot q}{n \mp 1 \cdot p} = \sqrt{\frac{4p}{n \pm 3}} = \sqrt[3]{\frac{4q}{n \mp 1}}$ , and the other two roots  $= -\frac{1}{2}r \cdot 1 \pm \sqrt{\pm n}$ , it follows, that if from the equation  $\sqrt{\frac{4p}{n \pm 3}} = \sqrt[3]{\frac{4q}{n \mp 1}}$ , where the two denominators under the radicals differ by 4, we can assign the value of  $n$ , the above formula will give us the roots.

38. As if the equation be  $x^3 - 18x = -27$ . Here  $p = 18$ , and  $q = 27$ ; then  $\sqrt{\frac{4p}{8}} = \sqrt{\frac{p}{2}} = \sqrt{9} = 3$ , and  $\sqrt[3]{\frac{4q}{4}} = \sqrt[3]{27} = 3$  also; therefore  $n + 3 = 8$ , or  $n - 1 = 4$ , either of which gives  $n = 5$ : consequently,  $r = \frac{n + 3 \cdot q}{n - 1 \cdot p} = \frac{8q}{4p} = \frac{2q}{p} = \frac{54}{18} = 3$  is the middle root, because  $\frac{8q}{4p}$  is found between the 4th and 12th cases, which are the limits of the middle roots: and  $-\frac{1}{2}r \cdot 1 \pm \sqrt{n} = -\frac{3}{2} \cdot 1 \pm \sqrt{5} = 4.854102$  and  $1.854102$  are the greatest and least roots. Or, these two roots may be also found in the same manner



manner from the table of forms, which contains all the roots of every equation, thus: by a few trials I find  $\sqrt[3]{\frac{4p}{20.95}} = \sqrt[3]{\frac{4q}{16.95}}$  nearly, and therefore  $\frac{20.95q}{16.95p} = 1.854$  is the least root, because here  $n = 17.95$  which lies far above the limit for the least roots, which is at the fourth form, where  $n$  is  $= 9$ . And lastly,  $\sqrt[3]{\frac{4p}{3.0557}} = \sqrt[3]{\frac{4q}{.9143}}$  nearly, and therefore,  $\frac{3.0557q}{.9143p} = 4.854$  is the greatest root, because  $\frac{3.0557q}{.9143p}$  is found between the 12th and 13th forms, which are the limits between which lies the greatest root of every equation that has all its roots real.

39. Again, let the equation be  $x^3 + 2x = 12$ . Here  $p = 2$ , and  $q = 12$ ; hence  $\sqrt[3]{\frac{4p}{2}} = \sqrt[3]{2p} = \sqrt[3]{4} = 2$ , and  $\sqrt[3]{\frac{4q}{6}} = \sqrt[3]{\frac{2}{3}q} = \sqrt[3]{8} = 2$  also; therefore  $n - 3 = 2$ , or  $n + 1 = 6$ , either of which gives  $n = 5$ . Consequently,  $r = \frac{n-3 \cdot q}{n+1 \cdot p} = \frac{2q}{6p} = \frac{q}{3p} = \frac{12}{6} = 2$ , and the other two roots are  $-\frac{1}{2}r \cdot 1 \pm \sqrt{-n} = -1 \cdot 1 \pm \sqrt{-5} = -1 \mp \sqrt{-5}$ .

40. But it is only by trials that we find out a proper value for  $n$  in such cases as these; and this is perhaps attended with no less trouble than the searching out one of the roots by trials from the original cubic equation itself. This method of finding the roots would indeed be effectual and satisfactory if we had a direct method of deter-

determining the value of  $n$  from the equation  $\sqrt[n]{\frac{4p}{n \pm 3}} = \sqrt[n]{\frac{4q}{n \mp 1}}$  by an equation under the 3d degree; but by reducing this equation out of radicals, there results another cubic equation of no less difficulty to resolve than the original one. We must therefore search for other methods of determining the roots; and first it will be proper to treat of the rule which is called CARDAN'S.

41. Let  $x^3 + px = q$  be the general equation where  $p$  and  $q$  denote any given numbers with their signs, positive or negative. And let  $z + y$  denote one of the roots of this equation, that is, let the root be divided into any two parts  $z$  and  $y$ . Hence then  $x = z + y$ ; which value of  $x$  being substituted for it in the original equation  $x^3 + px = q$ , that equation will become  $z^3 + 3z^2y + 3zy^2 + y^3 + p \cdot z + y = q$ , or  $z^3 + y^3 + 3zy \cdot z + y + p \cdot z + y = q$ . Now on introducing the two unknown quantities  $z$  and  $y$ , we supposed only one condition or equation, namely,  $z + y = x$ ; we are therefore yet at liberty to assume any other possible condition we please: but this other condition ought to be such as will make the equation reducible to a simple one, or to a quadratic, in order to obtain from it the value of  $z$  or  $y$ : and for this purpose there does not seem to be any other proper condition beside

that which supposes  $3xy$  to be  $= -p$ ; and in consequence of this supposition, the equation becomes barely  $x^3 + y^3 = q$ . Now from the square of this equation let four times the cube of  $xy = -\frac{1}{3}p$  be subtracted, and there will remain  $x^6 - 2x^3y^3 + y^6 = q^2 + \frac{4}{27}p^3$ , the square root of which is  $x^3 - y^3 = \sqrt{q^2 + \frac{4}{27}p^3}$ ; this last being added to, and subtracted from, the equation  $x^3 + y^3 = q$ ,

we have 
$$\begin{cases} 2x^3 = q + \sqrt{q^2 + \frac{4}{27}p^3} = q + 2\sqrt{\frac{1}{2}q^2 + \frac{1}{3}p^3}, \\ 2y^3 = q - \sqrt{q^2 + \frac{4}{27}p^3} = q - 2\sqrt{\frac{1}{2}q^2 + \frac{1}{3}p^3}, \end{cases}$$

hence dividing by 2, and extracting the cube roots, we

have 
$$\left\{ \begin{aligned} x &= \sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{2}q^2 + \frac{1}{3}p^3}} \times 1 \text{ or } \times -\frac{1 \pm \sqrt{-3}}{2} \\ y &= \sqrt[3]{\frac{1}{2}q - \sqrt{\frac{1}{2}q^2 + \frac{1}{3}p^3}} \times 1 \text{ or } \times -\frac{1 \mp \sqrt{-3}}{2} \end{aligned} \right\} \text{ the}$$

three values of  $x$  and  $y$ ; for every quantity has three different forms of the cube root, and the cube root of 1, is not only 1, but also  $-\frac{1 + \sqrt{-3}}{2}$  or  $-\frac{1 - \sqrt{-3}}{2}$ . Hence then the three values of  $x + y$  or  $x$ , or the three roots of the equation  $x^3 + px = q$ , are  $\sqrt[3]{\frac{1}{2}q + \sqrt{\frac{1}{2}q^2 + \frac{1}{3}p^3}} \times 1$  or  $\times -\frac{1 + \sqrt{-3}}{2}$  or  $\times -\frac{1 - \sqrt{-3}}{2} + \sqrt[3]{\frac{1}{2}q - \sqrt{\frac{1}{2}q^2 + \frac{1}{3}p^3}} \times 1$  or  $\times -\frac{1 - \sqrt{-3}}{2}$  or  $\times -\frac{1 + \sqrt{-3}}{2}$ , where the signs of  $\sqrt{-3}$  must be opposite in the values of  $x$  and  $y$ , that is, when it is  $\frac{1 \pm \sqrt{-3}}{2}$  in the one, it must be  $\frac{1 \mp \sqrt{-3}}{2}$  in the other,

otherwise

otherwise their product  $xy$  will not be  $= -\frac{1}{3}p$ , as it ought to be.

42. Or if we put  $a = \frac{1}{3}p$ , and  $b = \frac{1}{2}q$ , the same three roots will be

$$\begin{aligned} &\sqrt[3]{b + \sqrt{b^2 + a^3}} + \sqrt[3]{b - \sqrt{b^2 + a^3}} = \text{the 1st root or } r, \\ &-\frac{1}{2}\sqrt[3]{b + \sqrt{b^2 + a^3}} \cdot \sqrt{1 - \sqrt{-3}} - \frac{1}{2}\sqrt[3]{b - \sqrt{b^2 + a^3}} \cdot \sqrt{1 + \sqrt{-3}} \text{ the 2d root.} \\ &-\frac{1}{2}\sqrt[3]{b + \sqrt{b^2 + a^3}} \cdot \sqrt{1 + \sqrt{-3}} - \frac{1}{2}\sqrt[3]{b - \sqrt{b^2 + a^3}} \cdot \sqrt{1 - \sqrt{-3}} \text{ the 3d root.} \end{aligned}$$

43. Or again, the 1st root  $r$  being

$$\begin{aligned} &\sqrt[3]{b + \sqrt{b^2 + a^3}} + \sqrt[3]{b - \sqrt{b^2 + a^3}}, \text{ the other two will be} \\ &-\frac{1}{2}r + \frac{\sqrt{-3}}{2}\sqrt[3]{b + \sqrt{b^2 + a^3}} - \frac{\sqrt{-3}}{2}\sqrt[3]{b - \sqrt{b^2 + a^3}} = \text{the 2d root, and} \\ &-\frac{1}{2}r - \frac{\sqrt{-3}}{2}\sqrt[3]{b + \sqrt{b^2 + a^3}} + \frac{\sqrt{-3}}{2}\sqrt[3]{b - \sqrt{b^2 + a^3}} = \text{the 3d root.} \end{aligned}$$

44. Or, if we put  $s = \sqrt[3]{b + \sqrt{b^2 + a^3}}$ , and  $d = \sqrt[3]{b - \sqrt{b^2 + a^3}}$ , the roots will be

$$\begin{aligned} &s + d = r \text{ the 1st root,} \\ &-\frac{s+d}{2} + \frac{s-d}{2}\sqrt{-3} = \text{the 2d root,} \\ &-\frac{s+d}{2} - \frac{s-d}{2}\sqrt{-3} = \text{the 3d root.} \end{aligned}$$

45. The first of these roots  $x$  or  $r = s + d = \sqrt[3]{b + \sqrt{b^2 + a^3}} + \sqrt[3]{b - \sqrt{b^2 + a^3}}$ , is that which is called **CARDAN'S** rule, by whom it was first published, but invented by **FERREUS**. And this is always a real root, though it is not

always the greatest root as it has been commonly thought to be.

46. The first root  $r = s + d = \sqrt[3]{b + \sqrt{b^2 + a^3}} + \sqrt[3]{b - \sqrt{b^2 + a^3}}$ , although it be always a real quantity, yet often assumes an imaginary form when particular numbers are substituted instead of the letters  $a$  and  $b$ , or  $p$  and  $q$ . And this it is evident will happen whenever  $a$  is negative and  $a^3$  greater than  $b^2$ , or  $\sqrt[3]{p}$  greater than  $\frac{1}{2}q^{\frac{2}{3}}$ ; for then  $\sqrt{b^2 + a^3}$  becomes  $\sqrt{b^2 - a^3} = \sqrt{\frac{1}{2}q^{\frac{2}{3}} - \frac{1}{3}p^{\frac{2}{3}}}$  the square root of a negative quantity, which is imaginary. And this will evidently happen whenever the equation has three real roots, but at no time else, that is in all the first 13 cases of the foregoing table, wherein  $\sqrt[3]{p}$  is greater than  $\frac{1}{2}q^{\frac{2}{3}}$ , and  $p$  negative; the 4th and 13th only excepted, when  $\sqrt[3]{p}$  is  $= \frac{1}{2}q^{\frac{2}{3}}$ , and therefore  $\sqrt{b^2 - a^3} = 0$ , and two of the roots become equal, but with contrary signs. This root can never assume an imaginary form when  $a$  or  $p$  is positive, nor yet when  $p$  is negative and  $\frac{1}{2}q^{\frac{2}{3}}$  greater than  $\sqrt[3]{p}$ ; for in both these cases the quantity  $\sqrt{b^2 \pm a^3}$  is real, or the square root of a positive quantity. And these take place after the first 13 cases of the table of forms, that is, in all the cases which have only one real root. So that this rule of CARDAN's always gives the

root

root in an imaginary form when the equation has no imaginary roots, but in the form of a real quantity when it has imaginary roots.

47. It may, perhaps, seem wonderful that CARDAN'S theorem should thus exhibit the root of an equation under the form of an imaginary or impossible quantity always when the equation has no imaginary roots, but at no time else; and it may justly be demanded what can be the reason of so curious an accident. But this seeming paradox will be cleared up by the following consideration. It is plain, that this circumstance must have happened either through some impropriety in the manner of deducing the values of  $x$  and  $y$  from the two assumed equations  $x = z + y$ , and  $zy = -\frac{1}{3}p$ , or else by some impossibility in one of these two conditions themselves; but, on examination, the deductions are found to be all fairly drawn, and the operations rightly performed. The true cause must therefore lie concealed in one of these two conditions  $x = z + y$  and  $zy = -\frac{1}{3}p$ . In the first of them it cannot be, because it only supposes that a quantity  $x$  can be divided into two parts  $z$  and  $y$ , which is evidently a possible supposition: it can therefore nowhere exist but in the latter, namely,  $zy = -\frac{1}{3}p$ . Now this supposition is this, that the product of the two parts,  $z$  and  $y$ , into which the constant quantity  $x$  is divided, is

equal

equal to  $\frac{1}{3}p$  with its sign changed. But this may always take place when  $p$  is positive; for then  $-\frac{1}{3}p$  will be negative, and two numbers, the one positive and the other negative, may always be taken such that their product shall be equal to any negative number whatever, and yet their sum be equal to a given quantity  $x$ ; and this is done by taking the positive one as much greater than  $x$  as the other is negative; for thus it is evident the positive and negative numbers may be increased without end: wherefore there is no impossibility in the supposition when  $p$  is positive; and therefore then the formula ought to exhibit only real quantities, that is, in all the cases after the 16th in the table of forms, as we have before found. But the same thing cannot always happen when  $p$  is negative, or  $-\frac{1}{3}p = zy$  is positive: for that  $zy$  may be positive, the signs of the two factors  $z$  and  $y$  must be alike, either both  $+$  or both  $-$ , that is, both  $+$  when the sign of  $x$  is  $+$ , or both  $-$  when that is  $-$ : but it is well known, that the greatest product which can be made of the two parts into which a constant quantity  $x$  may be divided, is when the parts are equal to each other, or each equal  $\frac{1}{2}x$ , and therefore the greatest product is equal to  $\overline{\frac{1}{2}x}$ , or  $\frac{1}{4}x^2$ : wherefore if  $\frac{1}{4}x^2$  be equal to or greater than  $-\frac{1}{3}p$ , the condition which supposes that  $zy$  is  $= -\frac{1}{3}p$ , is possible, and the formula ought to express the root by real quantities

tities only, otherwise not; but  $\frac{1}{4}x^2$ , or  $\frac{1}{4}r^2$ , which is the same thing, is always less than  $-\frac{1}{3}p$  in the first thirteen cases of the table of forms; and therefore in all these cases, which are those in which  $\frac{1}{3}p^3$  is greater than  $\frac{1}{2}q^2$ , or all those which have three real roots, the formula ought to exhibit the root with imaginary quantities, as we have before found to happen; the 4th and 13th cases only excepted, in which  $\frac{1}{3}p^3$  is  $= \frac{1}{2}q^2$ , and therefore the quantity  $\sqrt{b^2 - a^3}$  vanishes, and two of the roots are equal.

48. Thus then the real cause of this circumstance is made manifest, and it is found to be the necessary consequence of the arbitrary hypothesis which was made, which is found to be possible only in certain cases. So that we cannot expect the formula to exhibit a real quantity in the other cases, since an impossible hypothesis must needs lead to an absurd conclusion.

49. The other two roots  $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3}$  in their general state appear in an imaginary form; but on the substitution of numbers for the letters in any example, they come out real or imaginary quantities in those cases in which they ought to be such. For  $s$  being  $= g + \sqrt{\mp b}$ , and  $d = g - \sqrt{\mp b}$  according as the roots are all real or only one is such; and  $-\frac{s+d}{2} = -g = -\frac{1}{2}r$  always half the one real root, we shall have  $\frac{s-d}{2} = \sqrt{\mp b}$  according to



the said two cases; and consequently  $\sqrt[3]{\frac{-d}{2}} \sqrt{-3} = \sqrt{\pm 3b}$  a real or an imaginary quantity according as the roots are to be real or imaginary.

50. The first root  $r$  being found from the formula  $\sqrt[3]{b + \sqrt{b^2 + a^3}} + \sqrt[3]{b - \sqrt{b^2 + a^3}}$ , or by any other means, the other two roots may be exhibited in several other forms besides the foregoing, as may be shewn in the following manner.

51. The equation being  $x^3 + px = q$ , and one root  $r$ , by substitution we have  $r^3 + pr = q$ , and, by subtracting, it is  $x^3 - r^3 + p \cdot \overline{x - r} = 0$ , and, dividing by  $x - r$ , it becomes  $x^2 + rx + r^2 + p = 0$ .

Or this same equation may be found by barely dividing  $x^3 + px - q = 0$  by  $x - r = 0$ , for the quotient is  $x^2 + rx + r^2 + p = 0$ . And the resolution of this quadratic equation gives  $x = -\frac{1}{2}r \pm \sqrt{-p - \frac{3}{4}r^2} = -\frac{1}{2}r \pm \frac{1}{2}\sqrt{-4p - 3r^2}$  the other two roots. And from hence again it appears, that these two roots are always imaginary when  $p$  in the given equation is positive; as also when it is negative and less than  $\frac{3}{4}r^2$ ; which again include all the cases of the table of forms after the 13th.

52. Again, since  $r^3 + pr = q$ , therefore  $r^2 + p = \frac{q}{r}$ , and  $r^2 = -p + \frac{q}{r}$ , and  $-3r^2 = 3p - \frac{3q}{r}$ ; which being

substituted in the above value of the two roots, they become  $-\frac{1}{2}r \pm \frac{1}{2}\sqrt{-p - \frac{3q}{r}}$ .

53. And again, if  $-p$  be expelled from this last form by means of its value  $r^2 - \frac{q}{r}$ , the same two roots will be expressed by  $-\frac{1}{2}r \pm \frac{1}{2}\sqrt{r^2 - \frac{4q}{r}} = -\frac{1}{2}r \times 1 \pm \sqrt{1 - \frac{4q}{r^3}}$ .

54. And farther, if  $r^3$  be expelled from this last form by means of its value  $q - pr$ , the same two roots will also become  $-\frac{1}{2}r \times 1 \pm \sqrt{1 - \frac{4q}{q - pr}} = -\frac{1}{2}r \times 1 \pm \sqrt{\frac{pr + 3q}{pr - q}}$ .

55. We might have derived the above forms in yet another manner thus. The first root being  $r$ , let the other two roots be  $v$  and  $w$ : then we shall have these two equations, namely,  $v + w = -r$ , and  $vwr = q$ , or  $vw = \frac{q}{r}$ ; from the square of the first of these subtract four times the last, so shall  $v^2 - 2vw + w^2 = r^2 - \frac{4q}{r}$ ; the root of this is  $v - w = \sqrt{r^2 - \frac{4q}{r}}$ , which being added to, and taken from  $v + w = -r$ , and dividing by 2, we have  $\frac{v}{w} = -\frac{1}{2}r \pm \frac{1}{2}\sqrt{r^2 - \frac{4q}{r}} = -\frac{1}{2}r \times 1 \pm \sqrt{1 - \frac{4q}{r^3}}$ , the same with one of the formulæ above given; and then by substitution the others will be deduced.

56. To illustrate now the rules  $x = s + d$ , or  $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3}$ , by some examples; suppose the

given equation to be  $x^3 - 36x = 91$ . Here  $p = -36$ ,  $q = 91$ ,  $a = \frac{1}{3}p = -12$ ,  $b = \frac{91}{2}$ ; then  $c = \sqrt{b^2 + a^3} = \sqrt{\frac{8281}{4} - 1728} = \sqrt{\frac{1369}{4}} = \frac{37}{2}$ ,  $s = \sqrt[3]{b + c} = \sqrt[3]{\frac{91}{2} + \frac{37}{2}} = \sqrt[3]{64} = 4$ , and  $d = \sqrt[3]{b - c} = \sqrt[3]{\frac{91}{2} - \frac{37}{2}} = \sqrt[3]{27} = 3$ . Consequently,  $r = s + d = 4 + 3 = 7$  the first root; and  $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = \frac{-7 \pm \sqrt{-3}}{2}$  the other two roots, which are imaginary.

57. Ex. 2. Let the equation be  $x^3 + 30x = 117$ . Here  $a = \frac{1}{3}p = 10$ ,  $b = \frac{1}{2}q = \frac{117}{2}$ ; then  $c = \sqrt{b^2 + a^3} = \sqrt{\frac{13689}{4} + 1000} = \sqrt{\frac{17689}{4}} = \frac{133}{2}$ ,  $s = \sqrt[3]{b + c} = \sqrt[3]{\frac{117}{2} + \frac{133}{2}} = \sqrt[3]{\frac{250}{2}} = \sqrt[3]{125} = 5$ , and  $d = \sqrt[3]{b - c} = \sqrt[3]{\frac{117}{2} - \frac{133}{2}} = \sqrt[3]{-\frac{16}{2}} = \sqrt[3]{-8} = -2$ . Consequently,  $r = s + d = 5 - 2 = 3$  the first root; and  $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = \frac{-3 \pm \sqrt{-3}}{2}$  the other two roots, which are imaginary.

58. Ex. 3. If the equation be  $x^3 + 18x = 6$ , we shall have  $a = 6$ , and  $b = 3$ ; then  $c = \sqrt{b^2 + a^3} = \sqrt{9 + 216} = \sqrt{225} = 15$ ,  $s = \sqrt[3]{b + c} = \sqrt[3]{3 + 15} = \sqrt[3]{18}$ , and  $d = \sqrt[3]{b - c} = \sqrt[3]{3 - 15} = \sqrt[3]{-12} = -\sqrt[3]{12}$ . Therefore  $r = s + d = \sqrt[3]{18} - \sqrt[3]{12} = .331313$  the first root; and  $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = -\frac{\sqrt[3]{18} - \sqrt[3]{12}}{2} \pm \frac{\sqrt[3]{18} + \sqrt[3]{12}}{2}\sqrt{-3}$  the other two roots.

59. Ex. 4. In the equation  $x^3 - 15x = 4$ , we have  $a = -5$ ,  $b = 2$ ; hence  $c = \sqrt{b^2 + a^3} = \sqrt{4 - 125} = \sqrt{-121}$

$$\sqrt{-121} = 11\sqrt{-1}, \quad s = \sqrt[3]{b+c} = \sqrt[3]{2+11\sqrt{-1}} = 2+\sqrt{-1}, \text{ and } d = \sqrt[3]{b-c} = \sqrt[3]{2-11\sqrt{-1}} = 2-\sqrt{-1}.$$

Wherefore  $r = s + d = 4$  the first root; and

$$-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = -2 \pm \sqrt{-1} \cdot \sqrt{-3} = -2 \pm \sqrt{3}$$

the other two roots, which are also real.

60. Ex. 5. The equation  $x^3 - 6x = 4$  gives  $a = -2$ , and  $b = 2$ ; therefore  $c = \sqrt{b^2 + a^3} = \sqrt{4 - 8} = \sqrt{-4} = 2\sqrt{-1}$ ,  $s = \sqrt[3]{b+c} = \sqrt[3]{2+2\sqrt{-1}} = -1 + \sqrt{-1}$ , and  $d = \sqrt[3]{b-c} = \sqrt[3]{2-2\sqrt{-1}} = -1 - \sqrt{-1}$ . And hence

$r = s + d = -2$  the first root; and

$$-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = 1 \pm \sqrt{-1} \cdot \sqrt{-3} = 1 \pm \sqrt{3} \text{ which}$$

are the two extremes, or the greatest and least roots. So that in this example, CARDAN's rule gives the middle root.

61. Ex. 6. Let the equation be  $x^3 - 9x = -10$ . Then  $a = -3$  and  $b = -5$ ; so that  $c = \sqrt{b^2 + a^3} = \sqrt{25 - 27} = \sqrt{-2}$ ,  $s = \sqrt[3]{b+c} = \sqrt[3]{-5+\sqrt{-2}} = 1 + \sqrt{-2}$ , and  $d = \sqrt[3]{b-c} = \sqrt[3]{-5-\sqrt{-2}} = 1 - \sqrt{-2}$ .

Hence  $r = s + d = 2$  the middle root; and

$$-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = -1 \pm \sqrt{-2} \cdot \sqrt{-3} = -1 \pm \sqrt{6}$$

the greatest and least roots.

62. Ex. 7. Take the equation  $x^3 - 12x = 9$ . Here  $a = -4$ , and  $b = \frac{9}{2}$ ; therefore  $c = \sqrt{b^2 + a^3} = \sqrt{\frac{81}{4} - 64}$

$= \sqrt{-\frac{175}{4}} = \frac{5}{2}\sqrt{-7}$ ,  $s = \sqrt[3]{b+c} = \sqrt[3]{\frac{9}{2} + \frac{5}{2}\sqrt{-7}} =$   
 $-\frac{3}{2} + \frac{1}{2}\sqrt{-7}$ , and  $d = \sqrt[3]{b-c} = \sqrt[3]{\frac{9}{2} - \frac{5}{2}\sqrt{-7}} =$   
 $-\frac{3}{2} - \frac{1}{2}\sqrt{-7}$ . Hence  $r = s + d = -3$  the middle root;  
 and  $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = \frac{3}{2} \pm \frac{1}{2}\sqrt{-7} \cdot \sqrt{-3} = \frac{3 \pm \sqrt{21}}{2}$   
 the greatest and least roots.

63. Ex. 8. Again, from the equation  $x^3 - 12x =$   
 $-8\sqrt{2}$ , we have  $a = -4$ , and  $b = -4\sqrt{2}$ ; hence  $c =$   
 $\sqrt{b^2 + a^3} = \sqrt{32 - 64} = \sqrt{-32} = 4\sqrt{-2}$ ,  $s = \sqrt[3]{b+c} =$   
 $\sqrt[3]{-4\sqrt{2} + 4\sqrt{-2}} = \sqrt{2} + \sqrt{-2}$ , and  $d = \sqrt{2} - \sqrt{-2}$ .  
 So that  $r = s + d = 2\sqrt{2}$  the middle root; and  
 $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = -\sqrt{2} \pm \sqrt{-2} \cdot \sqrt{-3} =$   
 $-\sqrt{2} \pm \sqrt{6} = -\sqrt{2} \cdot 1 \mp \sqrt{3}$  the greatest and least roots.

64. Ex. 9. But the equation  $x^3 - 15x = 22$  gives  
 $a = -5$ , and  $b = 11$ ; and therefore  $c = \sqrt{b^2 + a^3}$   
 $= \sqrt{121 - 125} = \sqrt{-4}$ ,  $s = \sqrt[3]{b+c} = \sqrt[3]{11 + \sqrt{-4}} =$   
 $-1 - \sqrt{-4}$ , and  $d = -1 + \sqrt{-4}$ . Consequently  
 $r = s + d = -2$  the least root; and  $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} =$   
 $1 \pm \sqrt{-4} \cdot \sqrt{-3} = 1 \pm \sqrt{12}$  the two greater roots.

65. Ex. 10. Lastly, in the equation  $x^3 - 15x = 20$ ,  
 we have  $a = -5$ , and  $b = 10$ ; consequently  $c = \sqrt{b^2 + a^3}$   
 $= \sqrt{100 - 125} = \sqrt{-25} = 5\sqrt{-1}$ ,  $s = \sqrt[3]{b+c} =$   
 $\sqrt[3]{10 + 5\sqrt{-1}}$ , and  $d = \sqrt[3]{10 - 5\sqrt{-1}}$ . Therefore

$r =$

$x = s + d = \sqrt[3]{10 + 5\sqrt{-1}} + \sqrt[3]{10 - 5\sqrt{-1}} =$  the first root; and  $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3} = -\frac{\sqrt[3]{10 + 5\sqrt{-1}} + \sqrt[3]{10 - 5\sqrt{-1}}}{2} \pm \frac{\sqrt[3]{10 + 5\sqrt{-1}} - \sqrt[3]{10 - 5\sqrt{-1}}}{2}\sqrt{-3} =$  the other two roots.

66. Hence it appears, that CARDAN's rule  $s + d$  brings out sometimes the greatest root, sometimes the middle root, and sometimes the least root.

*Of the Roots by Infinite Series.*

67. Another way of assigning the roots of a cubic equation, may be by infinite series, derived from the foregoing formulæ, namely,  $s + d$  and  $-\frac{s+d}{2} \pm \frac{s-d}{2}\sqrt{-3}$ , or

$$\sqrt[3]{b+c} + \sqrt[3]{b-c} \text{ and } -\frac{1}{2} \times \sqrt[3]{b+c} + \sqrt[3]{b-c} \pm \frac{1}{2}\sqrt{-3} \times \sqrt[3]{b+c} - \sqrt[3]{b-c}.$$

For by expanding  $\sqrt[3]{b \pm c}$  in an infinite series, we shall evidently have all the roots expressed in such series.

68. Now  $s = \sqrt[3]{b+c} = \sqrt[3]{b} \times : 1 + \frac{c}{3b} - \frac{2c^2}{3 \cdot 6b^2} + \frac{2 \cdot 5c^3}{3 \cdot 6 \cdot 9b^3} \&c.$   
 and  $d = \sqrt[3]{b-c} = \sqrt[3]{b} \times : 1 - \frac{c}{3b} - \frac{2c^2}{3 \cdot 6b^2} - \frac{2 \cdot 5c^3}{3 \cdot 6 \cdot 9b^3} \&c.$   
 Hence  $s + d = 2\sqrt[3]{b} \times : 1 - \frac{2c^2}{3 \cdot 6b^2} - \frac{2 \cdot 5 \cdot 8c^4}{3 \cdot 6 \cdot 9 \cdot 12b^4} \&c.$

for the first root, as it was found by Mr. NICOLE, in the

*Memoires de l'Acad.* 1738. Also

$$s-d = \frac{2c}{\sqrt[3]{b^2}} \times \frac{1}{3} + \frac{2 \cdot 5c^2}{3 \cdot 6 \cdot 9b^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11c^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15b^4} \&c. \text{ Therefore,}$$

$$\left. \begin{aligned} & - \frac{s+d}{2} \\ & \pm \frac{s-d}{2} \sqrt{-3} \end{aligned} \right\} = \left\{ \begin{aligned} & - \sqrt[3]{b} \times \frac{1}{3} - \frac{2c^2}{3 \cdot 6b^2} - \frac{2 \cdot 5 \cdot 8c^4}{3 \cdot 6 \cdot 9 \cdot 12b^4} \&c. \\ & \pm \frac{c \sqrt{-3}}{\sqrt[3]{b^2}} \times \frac{1}{3} + \frac{2 \cdot 5c^2}{3 \cdot 6 \cdot 9b} + \frac{2 \cdot 5 \cdot 8 \cdot 11c^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15b^4} \&c. \end{aligned} \right.$$

for the other two roots, which were given by CLAIRAUT, in his *Elemens d'Algebre*.

69. Hence again it appears, that when  $c^2$  is positive, these two latter roots are imaginary; for then the factor  $\frac{c\sqrt{-3}}{\sqrt[3]{b^2}}$  is imaginary. And that those roots are real when

this  $c^2$  is negative; for then this factor becomes

$$\frac{c\sqrt{-1} \times \sqrt{-3}}{\sqrt[3]{b^2}} = \frac{c\sqrt{3}}{\sqrt[3]{b^2}}, \text{ a real quantity. But in this last case,}$$

the sign of every second term in the two series must be changed, namely, the signs of the terms containing the odd powers of the negative quantity  $c^2$ ; for the series contain the letters as adapted to the positive sign only.

70. These series are proper for those cases only in which  $c^2$  is not greater than  $b^2$ ; for if  $c^2$  were greater than  $b^2$ , they would all diverge, and be of no use: and the series proper for the other cases, namely, in which  $c^2$  is greater than  $b^2$ , we shall give below.

71. That  $c^2$  be less than  $b^2$ , or the foregoing series be proper to be used,  $a$  or  $\frac{1}{3}b$  must be a negative quantity; for if it be positive, then  $c^2 = b^2 + a^3$  will be greater than  $b^2$ . But for this purpose  $a$  cannot be *any* negative quantity taken at pleasure; for if it be so taken as that  $a^3$  be greater than  $2.b^2$ , then shall  $-c^2 = a^3 - b^2$  be greater than  $b^2$ . And hence these series converge only in some of the cases of three real roots, and in some of those that have only one real root, namely, from the 16th form to somewhere between the 12th and 13th forms in the general table Art. 30. when  $b$  is positive, and consequently it includes some cases both with and without imaginary roots. But that in all the cases, the first series

$s + d = 2\sqrt[3]{b} \times : 1 - \frac{2c^2}{3.6b^2} \&c.$  is the greatest root, as will still more fully appear by consulting Art. 83.

72. Now, in the first place, when  $a = 0$ , or  $c = b$ , which is the limit, or 16th case in the table Art. 30, the equation being  $x^3 = q = 2b$ , then the only real root is  $s = \sqrt[3]{b + 0} = \sqrt[3]{2b} = \sqrt[3]{q} = \sqrt[3]{b} \times : 1 + \frac{1}{3} - \frac{2}{3.6} + \&c.$  Hence also, dividing by  $\sqrt[3]{b}$ , we have

$$\sqrt[3]{2} = 1 + \frac{1}{3} - \frac{2}{3.6} + \frac{2.5}{3.6.9} - \frac{2.5.8}{3.6.9.12} \&c.$$

73. But in this case also the root is

$$s + d = 2\sqrt[3]{b} \times : 1 - \frac{2}{3.6} - \frac{2.5.8}{3.6.9.12} \&c. \quad \text{And consequently}$$



consequently this is equal to the former series, or

$$2 \times : 1 - \frac{2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} \&c. = 1 + \frac{1}{3} - \frac{2}{3 \cdot 6} + \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} \&c.$$

$$= \sqrt[3]{2}. \text{ Hence, by subtracting } 1 - \frac{2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} \&c.$$

from both sides, we have

$$1 - \frac{2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} \&c. = \frac{1}{3} + \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} + \frac{2 \cdot 5 \cdot 8 \cdot 11}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15} \&c.$$

which multiplied by  $2\sqrt[3]{b}$  will also give the root of the same equation. And hence, adding  $\frac{2}{3 \cdot 6} + \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} \&c.$

to both sides of the last equation, we find that

$$1 \text{ is } = \frac{1}{3} + \frac{2}{3 \cdot 6} + \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} + \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} + \frac{2 \cdot 5 \cdot 8 \cdot 11}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15} \&c.$$

Or, farther, multiplying by 3, and subtracting 1, we have

$$2 = \frac{2}{6} + \frac{2 \cdot 5}{6 \cdot 9} + \frac{2 \cdot 5 \cdot 8}{6 \cdot 9 \cdot 12} + \frac{2 \cdot 5 \cdot 8 \cdot 11}{6 \cdot 9 \cdot 12 \cdot 15} \&c.$$

74. Also from  $2 \times : 1 - \frac{2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} \&c. = \sqrt[3]{2}$  in the last article, we find  $\frac{1}{2}\sqrt[3]{2} =$

$$1 - \frac{2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} \&c. = \frac{1}{3} + \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} + \frac{2 \cdot 5 \cdot 8 \cdot 11}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15} \&c.$$

75. In this case also, namely,  $c = b$ , the equation  $d = \sqrt[3]{b} - c = \sqrt[3]{b} \times : 1 - \frac{c}{\sqrt[3]{b}} - \frac{2c^2}{3b^{\frac{2}{3}}} \&c.$  becomes

$$0 = \sqrt[3]{b} \times : 1 - \frac{1}{3} - \frac{2}{3 \cdot 6} - \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} \&c.$$

And hence, dividing by  $\sqrt[3]{b}$ , and adding, we have

$$1 = \frac{1}{3} + \frac{2}{3 \cdot 6} + \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} + \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} \&c.$$

the same as in the last article but one.

76. And

76. And by taking other values of  $b$  and  $c$ , or other relations between them, any number of infinite series may be assigned, whose sums will be given by the two equations  $\sqrt[3]{b \pm c} = \sqrt[3]{b} \times 1 \pm \frac{c}{3b} - \frac{2c^2}{3 \cdot 6b^2} \pm \frac{2 \cdot 5c^3}{3 \cdot 6 \cdot 9b^3}$  &c. And if  $b$  be very great in respect of  $c$ , the two first terms of the series will give the cube root true to many places of figures.

77. Hitherto is concerning one of the limits or extreme cases only, namely, when  $c^2 = b^2$ , or when the equation is  $x^3 = q = 2b$ . And it has been observed, that the first general series for the three roots converges in all the cases of the equation  $x^3 - px = q$ , or  $x^3 - 3ax = 2b$ , in which  $a^3$  is not greater than  $2b^2$ . But  $a^3$  may be any real quantity not greater than  $2b^2$ , and so it may be either less than, equal to, or greater than  $b^2$ .

78. When, in this equation,  $a^3$  is less than  $b^2$ , then  $c^2$  is positive, and less than  $b^2$ , and the first series gives the only real root without any change in the signs of the terms. And to this belongs all cases of the equation that can fall in between the 13th and 16th formulæ in the general table in Art. 30.

79. If  $a^3$  be  $= b^2$ , then  $c = 0$ , and the three first series give  $2\sqrt[3]{b} = \sqrt[3]{4q}$  for the greatest root, and  $-\sqrt[3]{b}$  for each of the less roots. The same as at the 13th form in the general table Art. 30.

80. When  $a^3$  is greater than  $b^2$ ,  $c^2$  will be negative, and then, changing the signs of the odd powers of  $c^2$ , the three general series will give the three roots of the equation, which will always be all real. In this class are two cases, namely, when  $c^2$  is less than  $b^2$ , and when they are equal, which is the limit; for when  $c^2$  becomes greater than  $b^2$ , the series diverge.

81. Now when  $a^3$  is between  $b^2$  and  $2b^2$ , then  $c^2$  is negative and less than  $b^2$ , and the general series give all the three real roots by changing the sign of every other term.

82. And when  $a^3 = 2b^2$ , then  $-c^2 = b^2$ , and the three roots become thus:

$2\sqrt[3]{b} \times : 1 + \frac{2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18} \&c.$  the first or greatest root,

and  $\left\{ \begin{array}{l} -\sqrt[3]{b} \times : 1 + \frac{2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18} \&c. \\ \pm \sqrt[3]{b} \times \sqrt{3} \times : \frac{1}{3} - \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} + \frac{2 \cdot 5 \cdot 8 \cdot 11}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15} \&c. \end{array} \right\}$

the two less roots.

83. The first of these 3 is the greatest root, because  $\sqrt[3]{b} \times : 1 + \frac{2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} \&c.$  is greater than  $\sqrt[3]{b} \times \sqrt{3} \times : \frac{1}{3} - \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} \&c.$  for  $1 + \frac{2}{3 \cdot 6} \&c.$  is greater than 1, and  $\sqrt{3} \times : \frac{1}{3} - \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} \&c. = \sqrt{\frac{1}{3}} \times : 1 - \frac{2 \cdot 5}{6 \cdot 9} \&c.$

is

is less than 1. So that in general the first series gives the greatest of the three roots.

84. But it is evident, that this case agrees with the 10th form in the table Art. 30; in which the middle root  $r$  is found to be  $\sqrt[3]{\frac{4q}{2}} = \sqrt[3]{2q} = -\sqrt[3]{4b} = -2\sqrt[3]{\frac{1}{2}b}$ , and the two other, or greatest and least roots, are  $-\frac{1}{2}r \times 1 \pm \sqrt{3} = \sqrt[3]{\frac{1}{2}b} \times 1 \pm \sqrt{3}$ .

85. Hence by a comparison of these two different forms of the same roots we find

$$\sqrt[3]{3+1} = 1 + \frac{2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18} \&c. = A,$$

$$\text{and } \frac{\sqrt{3-1}}{2\sqrt{2}} = \frac{1}{3} - \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} + \frac{2 \cdot 5 \cdot 8 \cdot 11}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15} - \&c. = B.$$

86. And by adding and subtracting these two, we find

$$\frac{\sqrt[3]{3}}{\sqrt[3]{2}} = 1 + \frac{1}{3} + \frac{2}{3 \cdot 6} - \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} + + - - \&c. \text{ and}$$

$$\frac{1}{\sqrt[3]{2}} = 1 - \frac{1}{3} + \frac{2}{3 \cdot 6} + \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} - + + - - \&c. = C.$$

87. Also, because  $\frac{\sqrt[3]{3+1}}{2\sqrt[3]{2}} \times \frac{\sqrt[3]{3-1}}{2\sqrt[3]{2}}$  is  $= \frac{1}{2\sqrt[3]{4}}$ , which is  $= \frac{1}{2} \times \frac{1}{\sqrt[3]{2}}$ ; therefore the mean proportional between

the two series A and B, is to the series C, as the side of a square is to its diagonal.

88. Moreover, to and from the two series A and B, adding and subtracting the two series in Art. 74.

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namely,

namely,  $\frac{1}{2} \sqrt[3]{2}$  or  $\frac{\sqrt[3]{2}}{2} = 1 - \frac{2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} \&c. =$   
 $\frac{1}{3} + \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} + \frac{2 \cdot 5 \cdot 8 \cdot 11}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15} \&c.$  we obtain the 4 following series :

$$\frac{\sqrt[3]{3} + \sqrt[3]{4} + 1}{4\sqrt[3]{2}} = 1 - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} - \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 17 \cdot 20}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 21 \cdot 24} \&c.$$

$$\frac{\sqrt[3]{3} + \sqrt[3]{4} - 1}{4\sqrt[3]{2}} = \frac{1}{3} + \frac{2 \cdot 5 \cdot 8 \cdot 11}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 17 \cdot 20 \cdot 23}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 21 \cdot 24 \cdot 27} \&c.$$

$$\frac{\sqrt[3]{3} - \sqrt[3]{4} + 1}{4\sqrt[3]{2}} = \frac{2}{3 \cdot 6} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18} \&c.$$

$$\frac{\sqrt[3]{3} - \sqrt[3]{4} - 1}{4\sqrt[3]{2}} = -\frac{2 \cdot 5}{3 \cdot 6 \cdot 9} - \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 17}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 21} \&c.$$

89. It also appears, that the series

$$1 - \frac{1}{3} + \frac{2}{3 \cdot 6} + \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} - \frac{2 \cdot 5 \cdot 8 \cdot 11}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15} \&c.$$

is the reciprocal of the series

$$1 + \frac{1}{3} - \frac{2}{3 \cdot 6} + \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} + \frac{2 \cdot 5 \cdot 8 \cdot 11}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15} \&c.$$

where the signs of the former series are found by changing the signs of every other pair of terms in the latter; namely, omitting the first term, change the signs of the 2d and 3d terms, then passing over the 4th and 5th terms, change the signs of the 6th and 7th; and so on. For, by Art 86. the former of these series is equal to  $\frac{1}{\sqrt[3]{2}}$ ; and, by Art. 72. the latter is equal to  $\sqrt[3]{2}$ .

90. Let us now consider the cases in which  $c^2$  is greater than  $b^2$ , which include all the cases not comprehended by the former, or in which  $c^2$  is not greater than  $b^2$ . And this, it is evident, will happen both when  $a$  is positive and when negative; namely when  $a$  is any positive quantity whatever, or when it is any negative quantity, and  $a^3$  greater than  $2b^2$ . And in these two classes,  $c^2$  will be positive or negative, according as  $a$  is positive or negative.

91. Now the series in this class will be found the same way as in the last, by only writing here the letter  $c$  before the letter  $b$ ; for then we shall have  $s = \sqrt[3]{c + b}$ , and  $d = \sqrt[3]{-c + b} = -\sqrt[3]{c - b}$ .

$$\text{Then } s = \sqrt[3]{c + b} = \sqrt[3]{c} \times : 1 + \frac{b}{3c} - \frac{2b^2}{3 \cdot 6c^2} + \frac{2 \cdot 5b^3}{3 \cdot 6 \cdot 9c^3} \&c.$$

$$\text{and } d = -\sqrt[3]{c - b} = \sqrt[3]{c} \times : -1 + \frac{b}{3c} + \frac{2b^2}{3 \cdot 6c^2} + \frac{2 \cdot 5b^3}{3 \cdot 6 \cdot 9c^3} \&c.$$

$$\text{Hence } s + d = \frac{2b}{\sqrt[3]{c^2}} \times : \frac{1}{3} + \frac{2 \cdot 5b^2}{3 \cdot 6 \cdot 9c^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11b^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15c^4} \&c. =$$

the 1st root, and was given by CLAIRAUT. And

$$\left. \begin{aligned} & - \frac{s+d}{2} \\ & \pm \frac{s-d}{2} \sqrt{-3} \end{aligned} \right\} = \left\{ \begin{aligned} & \frac{-b}{\sqrt[3]{c^2}} \times : \frac{1}{3} + \frac{2 \cdot 5b^2}{3 \cdot 6 \cdot 9c^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11b^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15c^4} \&c. \\ & \pm \sqrt[3]{c} \cdot \sqrt{-3} \times : 1 - \frac{2b^2}{3 \cdot 6c^2} - \frac{2 \cdot 5 \cdot 8b^4}{3 \cdot 6 \cdot 9 \cdot 12c^4} \&c. \end{aligned} \right.$$

for the other two roots, which, I believe, are new.

92. Here it again appears, that when  $c^2$  is positive, the two latter roots are imaginary; because then

$\sqrt[3]{c} \times \sqrt{-3}$  will be imaginary. But if  $c^2$  be negative, those roots will be both real; since  $\sqrt[3]{c} \times \sqrt{-3}$  then becomes  $\sqrt[3]{c} \cdot \sqrt{-1} \times \sqrt{-3} = \sqrt[3]{c} \times -\sqrt{-1} \times \sqrt{-3} = -\sqrt[3]{c} \times \sqrt{3}$ . The signs prefixed to the terms as above, take place when  $c^2$  is positive; but when  $c^2$  shall be negative, the signs of the terms containing the odd powers of it must be changed. And these series include all the cases in which the former ones failed by not converging. So that between them they comprehend all the cases of the general cubic equation  $x^3 \pm px = q$ , as they each reciprocally converge when the other diverges, but in no other case, except in the common class, in which  $c$  is  $= b$ , which happens at the two limits, namely, either when  $a$  is  $= 0$ , or when  $-a^3 = 2b^2$ : and then they both give the same roots. But in the other cases they give the contrary roots; namely, when  $c$  is less than  $b$ , the first series gives the greatest root; and when  $c$  is greater than  $b$ , the latter series gives the least root.

93. Now when  $a$  is any positive quantity, the first of these series gives the only real root, without any change in the signs of the terms; the other two being imaginary. And this includes all the cases after the 16th in the table in Art. 30.

94. When  $a$  is  $= 0$ , or the limit between positive and negative, as in the 16th form in Art. 30. then is  $c = b$ ,

and the only real root, or the first series, becomes  $2\sqrt[3]{b} \times : \frac{1}{3} + \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} + \&c.$  which is the same root as was before found in Art. 73. So that in this 16th case, both this series and the series in Art. 67. converge, and give the same and only real root.

95. When  $a$  becomes negative, then  $c^2$  becomes negative, and the roots all real. But in this case the series only begins to converge when  $-a^3 = 2b^2$ , for then  $-c^2$  becomes  $= b^2$ , and then, making the proper change in the signs of the terms, the three roots become

1st.  $-2\sqrt[3]{b} \times : \frac{1}{3} - \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} + \frac{2 \cdot 5 \cdot 8 \cdot 11}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15} \&c.$  the least root, and

$$\left\{ \begin{array}{l} + \sqrt[3]{b} \times : \frac{1}{3} - \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} + \frac{2 \cdot 5 \cdot 8 \cdot 11}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15} \&c. \\ \pm \sqrt[3]{b} \cdot \sqrt{3} \times : 1 + \frac{2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18} \&c. \end{array} \right.$$
 the two greater roots.

96. I have here said, that the first of these three roots is the least of them. To prove which, I assert, that  $\sqrt[3]{3} \times : 1 + \frac{2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} \&c.$  is greater than 3 times  $\frac{1}{3} - \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} \&c.$  for  $3 \times : \frac{1}{3} - \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} \&c. = 1 - \frac{2 \cdot 5}{6 \cdot 9} \&c.$  is less than 1, whereas  $1 + \frac{2}{3 \cdot 6} \&c.$  is greater than 1. Consequently, the less of the two latter roots, namely,

$\sqrt[3]{b}$



$\sqrt[3]{b} \cdot \sqrt{3} \times : 1 + \frac{2}{3 \cdot 6} \&c. - \sqrt[3]{b} \times : \frac{1}{3} - \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} \&c.$  is greater than the first root  $2\sqrt[3]{b} \times : \frac{1}{3} - \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} \&c.$  That is to say, here the first is the least of the three roots, while in the other class of series the first is the greatest root.

97. Hence, comparing the value of any one of the roots here found, with the value of the same root as found in Art. 82, we obtain the relation between the two series that are concerned in them, namely, that the series

$1 + \frac{2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18} \&c.$  is to the series  $\frac{1}{3} - \frac{2 \cdot 5}{3 \cdot 6 \cdot 9} + \frac{2 \cdot 5 \cdot 8 \cdot 11}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15} - \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 17}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 21} \&c.$  as  $\sqrt{3} + 1$  is to  $\sqrt{3} - 1$ , or as  $2 + \sqrt{3}$  to 1, or as 1 to  $2 - \sqrt{3}$ , which are all equal to the same ratio. And the same thing appears from Art. 85.

98. When  $-a^3$  becomes greater than  $2b^2$ ,  $-c^2$  is greater than  $b^2$ , and, by the proper change in the signs, the series for the roots in all cases of this kind become

1st.  $\frac{-2b}{\sqrt[3]{c^2}} \times : \frac{1}{3} - \frac{2 \cdot 5b^2}{3 \cdot 6 \cdot 9c^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11b^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15c^4} \&c.$  the least root.

and  $\left\{ \begin{array}{l} + \frac{b}{\sqrt[3]{c^2}} \times : \frac{1}{3} - \frac{2 \cdot 5b^2}{3 \cdot 6 \cdot 9c^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11b^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15c^4} \&c. \\ \pm \sqrt[3]{c} \cdot \sqrt{3} \times : 1 + \frac{2b^2}{3 \cdot 6c^2} - \frac{2 \cdot 5 \cdot 8b^4}{3 \cdot 6 \cdot 9 \cdot 12c^4} \&c. \end{array} \right\}$  the two greater roots.

99. Let us now illustrate all the foregoing series for the roots of cubic equations, by finding by means of them

them the roots of the equations already treated of in Art. 56, &c.

100. And first in the equation  $x^3 - 36x = 91$ . Here  $p = -36$ ,  $q = 91$ ,  $a = -12$ ,  $b = 45\frac{1}{2}$ ,  $c^2 = b^2 + a^3 = \overline{45\frac{1}{2}}^2 - 12^3 = 342\frac{1}{4}$ , which being positive and less than  $b^2$ , this case belongs to the series

$$2\sqrt[3]{b} \times : 1 - \frac{2c^2}{3 \cdot 6b^2} - \frac{2 \cdot 5 \cdot 8c^4}{3 \cdot 6 \cdot 9 \cdot 12b^4} - \&c. \text{ in Art. 68.}$$

Now  $\frac{c^2}{b^2} = \frac{1369}{8281} = \overline{\frac{37}{91}}^2 = \cdot 1653182$ . Then

$$A = + 1 \cdot 0000000$$

$$B = \frac{2c^2}{3 \cdot 6b^2} A = - \cdot 0183687$$

$$C = \frac{5 \cdot 8c^2}{9 \cdot 12b^2} B = - \quad 11247$$

$$D = \frac{11 \cdot 14c^2}{15 \cdot 18b^2} C = - \quad 1061$$

$$E = \frac{17 \cdot 20c^2}{21 \cdot 24b^2} D = - \quad 118$$

$$F = \frac{23 \cdot 26c^2}{27 \cdot 30b^2} E = - \quad 14$$

$$G = \frac{29 \cdot 32c^2}{33 \cdot 36b^2} F = - \quad 2$$

$$\text{sum of the terms} = \quad \cdot 9803871$$


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$$2$$

$$1 \cdot 9607742 - \log. 0 \cdot 2924275$$

$$\sqrt[3]{b} = \sqrt[3]{45 \cdot 5} - - - - - 0 \cdot 5526705$$


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$$\text{hence the only real root is } 7 - - - - - 0 \cdot 8450980$$

That is,  $x = 7$  is  $= 2\sqrt[3]{\frac{91}{2}} \times : 1 - \frac{2 \cdot 37^2}{3 \cdot 6 \cdot 91^2} - \frac{2 \cdot 5 \cdot 8 \cdot 37^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 91^4} - \&c.$

101. The other two roots are imaginary, and in Art. 56 they were found to be  $= \frac{-7 \pm \sqrt{-3}}{2}$ ; but by means of the series in Art. 68, they are here found to be  $\frac{-7}{2} \pm \frac{c\sqrt{-3}}{\sqrt[3]{b^2}} \times : \frac{1}{3} + \frac{2 \cdot 5c^2}{3 \cdot 6 \cdot 9b^2} + \&c.$

Consequently we obtain these following sums :

$$\frac{7}{2}\sqrt[3]{\frac{91}{2}} = 1 - \frac{2 \cdot 37^2}{3 \cdot 6 \cdot 91^2} - \frac{2 \cdot 5 \cdot 8 \cdot 37^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 91^4} - \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 37^6}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 91^6} \&c.$$

$$\frac{1}{37}\sqrt[3]{\frac{91^2}{2^3}} = \frac{1}{3} + \frac{2 \cdot 5 \cdot 37^2}{3 \cdot 6 \cdot 9 \cdot 91^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 37^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 91^4} \&c.$$

102. Ex. 2. In the equation  $x^3 + 30x = 117$ , we have  $a = \frac{1}{3}p = 10$ ,  $b = \frac{1}{2}q = \frac{117}{2} = 58\frac{1}{2}$ , and  $c^2 = b^2 + a^3 = \left(\frac{133}{2}\right)^2$ , which being positive, and greater than  $b^2$ , the proper series for this is that in Art. 91, namely,

$$x = \frac{2b}{\sqrt[3]{c^2}} \times : \frac{1}{3} + \frac{2 \cdot 5b^2}{3 \cdot 6 \cdot 9c^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11b^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15c^4} + \&c.$$

Now  $\frac{b^2}{c^2} = \left(\frac{117}{133}\right)^2 = .7738308$ . Hence

$$A = \frac{1}{3} = .333$$

$$B = \frac{2 \cdot 5b^2}{6 \cdot 9c^2} A = 48$$

$$\frac{2b}{\sqrt[3]{c^3}} = 7.128$$

$$C = \frac{8 \cdot 11b^2}{12 \cdot 15c^2} B = 18$$

series inverted 124

$$D = \frac{14 \cdot 17b^2}{18 \cdot 21c^2} C = 9$$

2851

$$E = \frac{20 \cdot 23b^2}{24 \cdot 27c^2} D = 5$$

142

$$F = \frac{26 \cdot 29b^2}{30 \cdot 33c^2} E = 3$$

7

the root  $x = 3.000$

$$G = \frac{32 \cdot 35b^2}{36 \cdot 39c^2} F = 2$$

$$H = \frac{38 \cdot 41b^2}{42 \cdot 45c^2} G = 1$$

$$I = \frac{44 \cdot 47b^2}{48 \cdot 51c^2} H = 1$$

$$K = \frac{50 \cdot 53b^2}{54 \cdot 57c^2} I = 1$$

sum of the terms = .421

That is,  $x = 3 = \frac{2 \cdot 117}{\sqrt[3]{2 \cdot 133^2}} x : \frac{1}{3} + \frac{2 \cdot 5 \cdot 117^2}{3 \cdot 6 \cdot 9 \cdot 133^2} + \&c.$

103. By the other series in the same article the two imaginary roots come out

$$= -\frac{3}{2} \pm \sqrt[3]{c} \cdot \sqrt{-3} \times : 1 - \frac{2b^2}{3 \cdot 6c^2} - \&c. \text{ which were before found in Art. 57 to be } -\frac{3}{2} \pm \frac{7}{2}\sqrt{-3}.$$

Consequently

$$\frac{7}{2}\sqrt[3]{\frac{2}{133}} = 1 - \frac{2 \cdot 117^2}{3 \cdot 6 \cdot 133^2} - \frac{2 \cdot 5 \cdot 8 \cdot 117^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 133^4} - \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 117^6}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 133^6} \&c.$$

$$\frac{1}{39}\sqrt[3]{\frac{133^2}{2^2}} = \frac{1}{3} + \frac{2 \cdot 5 \cdot 117^2}{3 \cdot 6 \cdot 9 \cdot 133^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 117^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 133^4} + \&c.$$

104. Ex. 3. In the equation  $x^3 + 18x = 6$ , we have  
 $a = 6$ ,  $b = 3$ ,  $c = \sqrt{9 + 216} = \sqrt{225} = 15$ , real and  
 greater than  $b$ , and therefore this case belongs to the same  
 series as the last example. Now  $\frac{b^2}{c^2} = \frac{9}{225} = \frac{1}{25} = .04$ , and  
 $\frac{2b}{\sqrt{c^2}} = \frac{6}{\sqrt{225}} = \sqrt[3]{\frac{24}{25}} = \frac{1}{5} \sqrt[3]{120} = \sqrt[3]{.96}$ . Then

$$A = \frac{1}{3} = .3333333$$

$$B = \frac{2 \cdot 5 b^2}{6 \cdot 9 c^2} A = 24692$$

$$C = \frac{8 \cdot 11 b^2}{12 \cdot 15 c^2} B = 483$$

$$D = \frac{14 \cdot 17 b^2}{18 \cdot 21 c^2} C = 12$$

$$\begin{array}{r} .3358520 - \log. \bar{1}.5261480 \\ \sqrt[3]{.96} - - - - \bar{1}.9940904 \\ \hline \end{array}$$

$$\text{the root } x = .3313130 - - \bar{1}.5202384$$

And then the two imaginary roots are

$$- \frac{.331313}{2} \pm \sqrt[3]{c} \cdot \sqrt{-3} \times : 1 - \frac{2b^2}{3 \cdot 6c^2} \&c.$$

105. But, in Art. 58, these three roots were found  
 to be  $\sqrt[3]{18} - \sqrt[3]{12}$ , and  $-\frac{\sqrt[3]{18} - \sqrt[3]{12}}{2} \pm \frac{\sqrt[3]{18} + \sqrt[3]{12}}{2} \sqrt{-3}$ .  
 Consequently we have

$$\frac{\sqrt[3]{18} + \sqrt[3]{12}}{2\sqrt[3]{15}} = 1 - \frac{2}{3 \cdot 6 \cdot 25^2} - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 25^4} - \&c.$$

$$\frac{\sqrt[3]{18} - \sqrt[3]{12}}{2} \sqrt[3]{\frac{25}{3}} = \frac{1}{3} + \frac{2 \cdot 5}{3 \cdot 6 \cdot 9 \cdot 25^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 25^4} + \&c.$$

106. Ex. 4. In the equation  $x^3 - 15x = 4$ , we have  $a = -5$ ,  $b = 2$ , and  $c = \sqrt{b^2 + a^3} = \sqrt{-121} = 11\sqrt{-1}$ , imaginary and greater than  $b$ , which belongs to the same series as the last 2 examples, but changing the sign where the odd powers of the negative quantity  $c^2$  is concerned, as in Art. 98.

Now  $\frac{b^2}{c^2} = \frac{2^2}{11^2} = \frac{4}{121}$ , and  $\frac{2b}{\sqrt[3]{c^2}} = \frac{4}{\sqrt[3]{121}} = \sqrt[3]{\frac{64}{121}}$ . Then

$A = \frac{1}{3} = .3333333$	+	$B = \frac{2 \cdot 5b^2}{6 \cdot 9c^2} A = .0020406$
$C = \frac{8 \cdot 19b^2}{12 \cdot 15c^2} B =$		$D = \frac{14 \cdot 17b^2}{18 \cdot 21c^2} C =$
<div style="border-top: 1px solid black; display: inline-block; width: 100px; text-align: right;">330</div>		<div style="border-top: 1px solid black; display: inline-block; width: 100px; text-align: right;">7</div>
<div style="border-top: 1px solid black; display: inline-block; width: 100px; text-align: right;">+ .3333663</div>		<div style="border-top: 1px solid black; display: inline-block; width: 100px; text-align: right;">- .0020413</div>
<div style="border-top: 1px solid black; display: inline-block; width: 100px; text-align: right;">- .0020413</div>		

the series = .3313250 - log.  $\overline{1.5202543}$

$\sqrt[3]{\frac{64}{121}} - - - \overline{2.9077982}$

the least root = - .2679492 - -  $\overline{1.4280525}$

107. To find the other roots by this method, we must sum the series  $\sqrt[3]{c} \cdot \sqrt{3} x : 1 + \frac{2b^2}{3 \cdot 6c^2} - 8xc$ . And as the terms of it are found by multiplying the terms A, B, c, &c. of the former by  $\frac{3}{1}$ ,  $\frac{9}{5}$ ,  $\frac{15}{11}$ ,  $\frac{21}{17}$ , &c. respectively, we shall therefore have

$\alpha =$

$$\begin{array}{r|l}
 \alpha = \frac{3}{1} A = 1 & \\
 \beta = \frac{9}{5} B = 0.0036731 & \gamma = \frac{15}{11} C = -0.0000450 \\
 \delta = \frac{21}{17} D = 8 & \\
 \hline
 & + 1.0036739 \\
 & - 0.0000450 \\
 \hline
 \end{array}$$

$$\begin{array}{rcl}
 \text{series} = + 1.0036289 & - \log. & 0.0015732 \\
 \sqrt[3]{11} & - & 0.3471309 \\
 \sqrt{3} & - & 0.2385606 \\
 \hline
 & \pm & 3.8660254 - 0.5872647
 \end{array}$$

$$\left. \begin{array}{l} \frac{2}{3} \text{ the least root} \\ \text{with a contr. sign} \end{array} \right\} + 0.1339746$$

$$\text{sum} + 4.0000000 \text{ greatest root}$$

$$\text{diff.} - 3.7320508 \text{ middle root.}$$

108. But the same 3 roots, found in Art. 59, are also 4, and  $-2 \pm \sqrt{3}$ ; which being compared with the series in this example, we find

$$\frac{1 + 2\sqrt{3}}{2\sqrt[3]{11}} = 1 + \frac{2 \cdot 2^3}{3 \cdot 6 \cdot 11^2} - \frac{2 \cdot 5 \cdot 8 \cdot 2^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 11^4} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 2^6}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 11^6} \&c.$$

$$\frac{2 - \sqrt{3}}{4} \sqrt[3]{121} = \frac{1}{3} - \frac{2 \cdot 5 \cdot 2^2}{3 \cdot 6 \cdot 9 \cdot 11^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 2^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 11^4} - \&c.$$

109. Ex. 5. In the equation  $x^3 - 6x = 4$ , we have  $a = -2$ ,  $b = 2$ , and  $c^2 = b^2 + a^3 = 4 - 8 = -4$ , which being negative, and  $\neq b^2$ , this case belongs to the series either in Art. 82 or 95. The operation of fudging the terms by them is here omitted, because so much

room

room would be necessary to set down so great a number of terms, and as the properties arising from the series in this case have already been noticed above. The 3 roots of this equation have been found in Art. 60 to be  $-2$  and  $1 \pm \sqrt{3}$ .

110. Ex. 6. In the equation  $x^3 - 9x = -10$ , we have  $a = -3$ ,  $b = -5$ , and  $c^2 = 25 - 27 = -2$ , which being negative and less than  $b^2$ , the general series in Art. 68, with the necessary change of the signs, will give the 3 roots. Now  $\frac{c^2}{b^2} = \frac{2}{25} = \frac{8}{100} = .08$ , and  $\sqrt[3]{b} = -\sqrt[3]{5}$ , also  $\frac{c\sqrt{-3}}{\sqrt[3]{b^2}} = \frac{\sqrt{6}}{\sqrt[3]{25}}$ . Hence

$$\begin{array}{rcl}
 A = & = 1 & \\
 B = \frac{3c^2}{3 \cdot 6b^2} A = 0.0088889 & \left| \begin{array}{l} C = \frac{5 \cdot 8c^2}{9 \cdot 12b^2} B = .0002634 \\ E = \frac{17 \cdot 20c^2}{21 \cdot 24b^2} D = 7 \end{array} \right. & \\
 D = \frac{11 \cdot 14c^2}{15 \cdot 18b^2} C = 120 & & \\
 & + 1.0089009 & \\
 & - 0.0002641 & \\
 & + 1.0086368 & \\
 & \underline{\quad\quad\quad} & \\
 & 2 & \\
 & \underline{\quad\quad\quad} & \\
 & 2.0172736 & - \log. 0.3047649 \\
 \sqrt[3]{b} = \sqrt[3]{5} & - - - & 0.2329900
 \end{array}$$

the greatest root =  $-3.44948974 - 0.5377549$

III. Then



111. Then for the other roots, by multiplying the terms A, B, C, &c. of the former by  $\frac{1}{3}$ ,  $\frac{5}{9}$ ,  $\frac{11}{15}$ , &c. we have

$$\begin{array}{r|l} \alpha = \frac{1}{3} A = .3333333 & \epsilon = \frac{5}{9} B = .0049383 \\ \gamma = \frac{11}{15} C = 1931 & \delta = \frac{17}{15} D = 97 \\ \varepsilon = \frac{23}{27} E = 6 & - .0049480 \end{array}$$

$$+ .3335270$$

$$- .0049480$$

$$.3285790 \quad - \quad - \quad - \quad \log. \quad 1.5166398$$

$$\frac{\sqrt{6}}{\sqrt{25}} \quad - \quad - \quad - \quad - \quad - \quad 1.9230956$$

$$\text{the second series} \quad \pm \quad .27525513 \quad - \quad - \quad 1.4397354$$

$$\frac{1}{2} \text{ the greatest root} + 1.72474487$$

$$\text{middle root} \quad 2.00000000$$

$$\text{least root} \quad 1.44948974$$

112. But, by Art. 61, these 3 roots were found to be 2 and  $-1 \pm 6$ ; which being compared with the series belonging to this case, we find

$$\frac{\sqrt{6+1}}{\sqrt[3]{5}} = 1 + \frac{2.2}{3.6.25} - \frac{2.5.8.2^2}{3.6.9.12.25^2} + \frac{2.5.8.11.14.2^3}{3.6.9.12.15.18.25^3} \&c.$$

$$\frac{\sqrt{6-2}}{4} \sqrt[3]{25} = \frac{1}{3} - \frac{2.5.2}{3.6.9.25} + \frac{2.5.8.11.2^2}{3.6.9.12.15.25^2} - \&c.$$

113. Ex. 7. In the equation  $x^3 - 12x = 9$ , we have  $a = -4$ ,  $b = \frac{9}{2}$ , and  $c^2 = \frac{81}{4} - 64 = -\frac{175}{4}$ , which being negative, and greater than  $b^2$ , we shall have 3 real roots by the series in Art. 98.

Now  $\frac{b^3}{c^3} = \frac{81}{175}$ ,  $\frac{b}{\sqrt[3]{c^2}} = \frac{9}{\sqrt[3]{350}} = \sqrt[3]{\frac{729}{350}}$ , and

$\sqrt[3]{c} = \sqrt[6]{\frac{175}{4}} = \sqrt[6]{43.75}$ . Then

$A = \frac{1}{3}$	$= .33333$	$B = \frac{2.5b^2}{6.9c^2}$	$A = .02857$
$C = \frac{8.11b^2}{12.15c^2}$	$B =$	$D = \frac{14.17b^2}{18.21c^2}$	$C =$
$E = \frac{20.23b^2}{24.27c^2}$	$D =$	$F = \frac{26.29b^2}{30.33c^2}$	$E =$
$G = \frac{32.35b^2}{36.39c^2}$	$F =$	$H = \frac{38.41b^2}{42.45c^2}$	$G =$
$I = \frac{44.47b^2}{48.51c^2}$	$H =$	$K = \frac{50.53b^2}{54.57c^2}$	$I =$

$$+ .34051$$

$$- .03071$$

$$.30980$$

$$2$$

$$.61960$$

$$\sqrt[3]{\frac{729}{350}}$$

$$- .03071$$

$$- - \log. \bar{1}.7921114$$

$$- - - 0.1062198$$

$$\text{the least root} = - .79128 - - \bar{1}.8983312$$

114. Then, since the terms of the latter series are found by multiplying the terms of the former by the fractions  $\frac{3}{1}$ ,  $\frac{9}{5}$ ,  $\frac{15}{17}$ ,  $\frac{21}{17}$ , &c. they will be thus :

$\alpha = \frac{3}{1} A =$	1'00000		$\gamma = \frac{15}{11} C =$	.00882
$\mathcal{E} = \frac{9}{5} B =$	5143		$\varepsilon = \frac{27}{23} E =$	73
$\delta = \frac{21}{17} D =$	232		$\eta = \frac{39}{35} G =$	9
$\zeta = \frac{33}{29} F =$	25		$\iota = \frac{51}{47} I =$	<u>I</u>
$\theta = \frac{45}{41} H =$	4			- .00965
	+ 1'05404			
	- 0'00965			
	<u>1'04439</u>			
	$\sqrt[6]{43'75}$	- - -	log.	0'0188627
	$\sqrt{3}$	- - -		0'2734963
		- - -		<u>0'2385606</u>
last series $\pm$	3'39564	- -		0'5309196
$-\frac{1}{2}$ the first	+ 0'39564			
greatest root	+ 3'79128			
middle root	- 3'00000			

115. But, by Art. 62, these same 3 roots are, - 3, and  $\frac{3 \pm \sqrt{21}}{2}$ ; which being compared with the series belonging to this case, we find

$$\frac{\sqrt[12]{21+9}}{\sqrt[12]{350}} \sqrt{6} = 1 + \frac{2 \cdot 81}{3 \cdot 6 \cdot 175} - \frac{2 \cdot 5 \cdot 8 \cdot 81^2}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 175^2} + \&c.$$

$$\frac{\sqrt[12]{21-3}}{36} \sqrt[3]{350} = \frac{1}{3} - \frac{2 \cdot 5 \cdot 81}{3 \cdot 6 \cdot 9 \cdot 175} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 81^2}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 175^2} - \&c.$$

116. Ex. 8. In the equation  $x^3 - 12x = -8\sqrt{2}$ , we have  $a = -4$ ,  $b = -4\sqrt{2}$ , and  $c^2 = 32 - 64 = -32$ , which being negative, and equal to  $b^2$ , the 3 roots will

be

be found, by both the forms of series, like as in Ex. 5, Art. 109; but the operation is here omitted for the same reasons as were there given. The 3 roots of this equation were, in Art. 63, found to be  $2\sqrt{2}$  and  $-\sqrt{2} \pm \sqrt{6}$ .

117. Ex. 9. In the equation  $x^3 - 15x = 22$ , we have  $a = -5$ ,  $b = 11$ , and  $c^2 = 121 - 125 = -4$ , which being negative, and less than  $b^2$ , the series in Art. 68 give these 3 roots:

$$\begin{aligned} \text{Greatest root} &= 2\sqrt[3]{11} \times : 1 + \frac{2c^2}{3 \cdot 6b^2} - \frac{2 \cdot 5 \cdot 8c^4}{3 \cdot 6 \cdot 9 \cdot 12b^4} \&c. \\ \text{The two less roots} &\left\{ \begin{aligned} &-\sqrt[3]{11} \times : 1 + \frac{2c^2}{3 \cdot 6b^2} - \frac{2 \cdot 5 \cdot 8c^4}{3 \cdot 6 \cdot 9 \cdot 12b^4} \&c. \\ &\pm \frac{2\sqrt{3}}{\sqrt[3]{121}} \times : \frac{1}{3} - \frac{2 \cdot 5c^2}{3 \cdot 6 \cdot 9b^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11c^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15b^4} \&c. \end{aligned} \right\} \text{where } \frac{c^2}{b^2} = \frac{4}{121} \end{aligned}$$

Here

$$\begin{array}{rcl} A = & = 1.0000000 & \\ B = \frac{2c^2}{3 \cdot 6b^2} A = & 36731 & C = \frac{5 \cdot 8c^2}{9 \cdot 12b^2} B = -0.0000450 \\ D = \frac{11 \cdot 14c^2}{15 \cdot 18b^2} C = & 8 & \\ & \hline & + 1.0036739 & \\ & - 0.0000450 & \\ & \hline & 1.0036289 & \\ & \hline & 2 & \\ & \hline & 2.0072578 & - \log. 0.3026031 \\ & \sqrt[3]{11} & - \quad \quad \quad 0.3471309 \\ & & \hline \text{the greatest root} = 4.4641016 & - \quad \quad \quad 0.6497340 \\ & \text{M m m 2} & \quad \quad \quad 118. \text{ Again,} \end{array}$$

118. Again,

$$\alpha = \frac{1}{3} A = \cdot 3333333 \quad \left| \quad \epsilon = \frac{5}{9} B = \cdot 0020406$$

$$\gamma = \frac{11}{15} C = \quad 330 \quad \left| \quad \delta = \frac{17}{21} D = \quad 7$$

$$+ \cdot 3333663 \quad - \cdot 0020413$$

$$- \cdot 0020413$$

$$\cdot 3313250$$

$$2$$

$$\cdot 6626500 \quad - \quad - \quad \log. \quad \overline{1} \cdot 8212842$$

$$\sqrt{3} \quad - \quad - \quad - \quad 0 \cdot 2385606$$

$$\sqrt[3]{121} \quad - \quad - \quad - \quad - \quad 0 \cdot 6942618$$

$$\text{the latter series} \pm \cdot 2320508 \quad - \quad - \quad \overline{1} \cdot 3655830$$

$$\frac{1}{2} \text{ the first} - \cdot 2320508$$

$$\text{middle root} = - \cdot 24641016$$

$$\text{least root} = - \cdot 20000000$$

119. But, by Art. 64, the 3 roots are  $-2$  and  $1 \pm \sqrt{12}$ ; hence

$$\frac{1+2\sqrt{3}}{2\sqrt{11}} = 1 + \frac{2 \cdot 2^2}{3 \cdot 6 \cdot 11^2} - \frac{2 \cdot 5 \cdot 8 \cdot 2^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 11^4} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 2^6}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 11^6} \&c.$$

$$\frac{2-\sqrt{3}}{4} \sqrt[3]{121} = \frac{1}{3} - \frac{2 \cdot 5 \cdot 2^2}{3 \cdot 6 \cdot 9 \cdot 11^3} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 2^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 11^4} - \&c.$$

120. And in this manner the roots of cubic equations may always be found by these series; and then by comparing them with the roots of the same equations, as found by other methods, we shall obtain as many series as we please, whose sums will be given.

121. Hence

121. Hence also we may find the sum of any general series of either of these forms, namely,

$$1 \mp \frac{2g^2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8g^4}{3 \cdot 6 \cdot 9 \cdot 12} \mp \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14g^6}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18} \&c. \text{ or}$$
  

$$\frac{1}{3} \pm \frac{2 \cdot 5g^2}{3 \cdot 6 \cdot 9} + \frac{2 \cdot 5 \cdot 8 \cdot 11g^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15} \pm \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 17g^6}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 21} \&c. \text{ by comparing them with the roots of given cubic equations; whatever be the value of } g, \text{ not greater than } 1.$$

122. For, by Art. 68,  $\sqrt[3]{b+c} + \sqrt[3]{b-c} = 2\sqrt[3]{b} \times :$   
 $1 - \frac{2c^2}{3 \cdot 6b^2} - \frac{2 \cdot 5 \cdot 8c^4}{3 \cdot 6 \cdot 9 \cdot 12b^4} \&c. \text{ is = the greatest root of the}$   
 cubic equation  $x^3 - 3\sqrt[3]{b^2-c^2} \cdot x = 2b$ . Now make  $2\sqrt[3]{b} = 1$ , and  $\frac{c^2}{b^2} = g^2$ ; so shall the above become  
 $\frac{1}{2}\sqrt[3]{1+g^2} + \frac{1}{2}\sqrt[3]{1-g^2} = 1 - \frac{2g^2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8g^4}{3 \cdot 6 \cdot 9 \cdot 12} \&c. = \text{the greatest}$   
 root of the equation  $x^3 - \frac{3}{4}\sqrt[3]{1-g^2} \cdot x = \frac{1}{4}$ . And when  $g^2$  or  $\frac{c^2}{b^2}$  is negative, these become

$\frac{1}{2}\sqrt[3]{1+g\sqrt{-1}} + \frac{1}{2}\sqrt[3]{1-g\sqrt{-1}} = 1 + \frac{2g^2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8g^4}{3 \cdot 6 \cdot 9 \cdot 12} + \&c.$   
 = the greatest root of the equation  $x^3 - \frac{3}{4}\sqrt[3]{1+g^2} \cdot x = \frac{1}{4}$ .

So that in general the infinite series

$1 \mp \frac{2g^2}{3 \cdot 6} - \frac{2 \cdot 5 \cdot 8g^4}{3 \cdot 6 \cdot 9 \cdot 12} \mp \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14g^6}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18} \&c. \text{ is}$   
 $= \frac{1}{2}\sqrt[3]{1+g\sqrt{\pm 1}} + \frac{1}{2}\sqrt[3]{1-g\sqrt{\pm 1}} = \text{the greatest root of the}$   
 equation  $x^3 - \frac{3}{4}\sqrt[3]{1 \mp g^2} \cdot x = \frac{1}{4}$ . Where the upper and under signs respectively correspond to each other.

123. Again,

123. Again,

$$\sqrt[3]{c+b} - \sqrt[3]{c-b} = \frac{2b}{\sqrt[3]{c^2}} \times \left\{ \frac{1}{3} + \frac{2 \cdot 5 b^2}{3 \cdot 6 \cdot 9 c^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11 b^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 c^4} \right\} \&c.$$

is the least root of the equation  $x^3 + 3\sqrt[3]{c^2 - b^2} \cdot x = 2b$ .

Then, by taking  $\frac{2b}{\sqrt[3]{c^2}} = 1$ , and  $\frac{b^2}{c^2} = g^2$ , this becomes

$$\frac{\sqrt[3]{1+g} - \sqrt[3]{1-g}}{2g} = \frac{1}{3} + \frac{2 \cdot 5 g^2}{3 \cdot 6 \cdot 9} \&c. = \text{the least root of the}$$

equation  $x^3 + \frac{3\sqrt[3]{1-g^2}}{4g^2} x = \frac{1}{4g^2}$ . And when  $g^2$  or  $c^2$  is negative, this becomes

$$\frac{\sqrt[3]{1+g\sqrt{-1}} - \sqrt[3]{1-g\sqrt{-1}}}{2g\sqrt{-1}} = \frac{1}{3} - \frac{2 \cdot 5 g^2}{3 \cdot 6 \cdot 9} + \&c. = \text{the least root}$$

of the equation  $x^3 - \frac{3\sqrt[3]{1+g^2}}{4g^2} x = \frac{-1}{4g^2}$ . So that in general the infinite series

$$\frac{1}{3} \pm \frac{2 \cdot 5 g^2}{3 \cdot 6 \cdot 9} + \frac{2 \cdot 5 \cdot 8 \cdot 11 g^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15} \pm \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 17 g^6}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 21} \&c. \text{ is}$$

$$= \frac{\sqrt[3]{1+g\sqrt{\pm 1}} - \sqrt[3]{1-g\sqrt{\pm 1}}}{2g\sqrt{\pm 1}} = \text{the least root of the equa-}$$

$$\text{tion } x^3 \pm \frac{3\sqrt[3]{1 \mp g^2}}{4g^2} x = \frac{\pm 1}{4g^2}.$$

### *Of the Roots by another Class of Series.*

124. But there are yet other series, converging much faster than those in the foregoing class, by the help of which, and CARDAN'S rule conjointly, may always be found

found the roots of those equations in which that rule fails when it is applied singly, that is, in what is called the irreducible case, or that in which  $c^2$  is negative. And those series are found by introducing another cubic equation having the same values of  $b$  and  $c^2$  as the given equation, except that in the new equation the value of  $c^2$  is positive, while in the given one it is negative. For when  $c^2$  is positive, the new equation to which it belongs has only one real root, and that root is always found by CARDAN'S rule; but the contrary takes place when  $c^2$  is negative, the equation having then three real roots, although they are not always determinable by that rule, because the radical quantities can seldom be extracted, on account of the square root of the negative quantity which is contained in them.

125. Now the general expression for the root by CARDAN'S rule being  $s + d = \sqrt[3]{b + \sqrt{\pm c^2}} + \sqrt[3]{b - \sqrt{\pm c^2}}$  or  $\sqrt[3]{\sqrt{\pm c^2} + b} - \sqrt[3]{\sqrt{\pm c^2} - b}$ , if the cubic roots of each of these be extracted by the binomial theorem, as at Art. 68, we shall obtain these 4 forms;

$$1. \sqrt[3]{b + \sqrt{+c^2}} + \sqrt[3]{b - \sqrt{+c^2}} = 2\sqrt[3]{b} \times : 1 - \frac{2c^2}{3 \cdot 6b^2} - 8xc.$$

$$2. \sqrt[3]{b + \sqrt{-c^2}} + \sqrt[3]{b - \sqrt{-c^2}} = 2\sqrt[3]{b} \times : 1 + \frac{2c^2}{3 \cdot 6b^2} - 8xc.$$

$$3. \sqrt[3]{\sqrt{+c^2} + b} - \sqrt[3]{\sqrt{+c^2} - b} = \frac{2b}{\sqrt[3]{c^2}} \times : \frac{1}{3} + \frac{2 \cdot 5b^2}{3 \cdot 6 \cdot 9c^2} + 8xc.$$

$$4. \sqrt[3]{\sqrt{-c^2} + b} - \sqrt[3]{\sqrt{-c^2} - b} = \frac{2b}{\sqrt[3]{c^2}} \times : -\frac{1}{3} + \frac{2 \cdot 5b^2}{3 \cdot 6 \cdot 9c^2} - 8xc.$$

126. Of



126. Of which the series in the first and third denote the only real root of the equation when  $c^2$  is positive, according as  $c$  is greater or less than  $b$ , which root call  $x$ ; and the series in the second and fourth forms denote the greatest and least roots of the equation when  $c^2$  is negative, which roots call  $R$  and  $r$  respectively. Then by adding and subtracting the first and second, as also the third and fourth, there result these four equations;

$$R + X = 4\sqrt[3]{b} \times : 1 - \frac{2.5.8c^4}{3.6.9.12b^4} - \frac{2.5.8.11.14.17.20c^8}{3.6.9.12.15.18.21.24b^8} \&c.$$

$$R - X = 4\sqrt[3]{b} \times : \frac{2c^2}{3.6b^2} + \frac{2.5.8.11.14c^6}{3.6.9.12.15.18b^6} + \&c.$$

$$X - r = \frac{4b}{\sqrt[3]{c^2}} \times : \frac{1}{3} + \frac{2.5.8.11b^4}{3.6.9.12.15c^4} + \&c.$$

$$X + r = \frac{4b}{\sqrt[3]{c^2}} \times : \frac{2.5b^2}{3.6.9c^2} + \frac{2.5.8.11.14.17b^6}{3.6.9.12.15.18.21c^6} + \&c.$$

127. And hence, by equal addition or subtraction, we find these two different expressions both for the greatest and least roots of a cubic equation in which  $c^2$  or  $b^2 + a^3$  is negative, namely,

$$R = -X + 4\sqrt[3]{b} \times : 1 - \frac{2.5.8c^4}{3.6.9.12b^4} - \frac{2.5.8.11.14.17.20c^8}{3.6.9.12.15.18.21.24b^8} \&c. \text{ or}$$

$$R = X + 4\sqrt[3]{b} \times : \frac{2c^2}{3.6b^2} + \frac{2.5.8.11.14c^6}{3.6.9.12.15.18b^6} + \&c.$$

$$r = X - \frac{4b}{\sqrt[3]{c^2}} \times : \frac{1}{3} + \frac{2.5.8.11b^4}{3.6.9.12.15c^4} + \frac{2.5.8.11.14.17.20.23b^8}{3.6.9.12.15.18.21.24.27c^8} \&c. \text{ or}$$

$$r = -X + \frac{4b}{\sqrt[3]{c^2}} \times : \frac{2.5b^2}{3.6.9c^2} + \frac{2.5.8.11.14.17b^6}{3.6.9.12.15.18.21c^6} + \&c.$$

where  $R$  is the greatest, and  $r$  the least root of the equation

tion  $x^3 - 3ax = 2b$  or  $x^3 - 3\sqrt[3]{c^2 + b^2} \cdot x = 2b$ , and  $x$  the only real root of the equation  $x^3 + 3\sqrt[3]{c^2 - b^2} \cdot x = 2b$ ; in which, as well as in the above series,  $c^2$  denotes a positive quantity.

128. And hence it can no longer be said that CARDAN'S rule is of no use in the solution of cubic equations that have three real roots; since they have here been reduced to the other case in which the equation has but one real root, which case is always resolvable by that rule. And the first hint of such reduction I received from FRANCIS MASERES, Esq. Curfitor Baron of the Exchequer, he having done me the favour to communicate to me the second of the above four forms for the greatest root, in a letter of the 17th of July 1779; the investigation of which formula, together with those of the other three, nearly as above, I had the honour of sending him in a letter of the 26th of the same month; and that learned gentleman has since communicated to the Royal Society his said formula, together with his own investigation of it, done in his usual very accurate manner. Since that time I have seen, in the *Memoires de l'Acad.* for the year 1743, four expressions similar to the above, given by Mr. NICOLE for the purpose of summing certain terms of a binomial raised to

any power, but unaccompanied with any appearance of the idea of thus reducing the one case of the cubic equation to the other.

129. It is hardly necessary to remark, that any general series of each of the above four forms, is summed by means of the sum or difference of the roots of these two equations  $x^3 - 3\sqrt{b^2 \pm c^2} \cdot x = 2b$ , and that by substituting particular numbers for  $b$  and  $c$ , we may thus sum as many series of those forms as we please.

130. Ex. 1. We may now illustrate these formulas by some examples. And first in the equation  $x^3 - 15x = 4$ . Here  $2b = 4$ , and  $3\sqrt{b^2 + c^2} = 15$ , consequently  $b = 2$ , and  $c^2 = 5^3 - b^2 = 125 - 4 = 121 = 11^2$ , and  $x = \sqrt[3]{c+b} - \sqrt[3]{c-b} = \sqrt[3]{13} - \sqrt[3]{9} = .2712508$  the root of the equation  $x^3 - 3\sqrt{b^2 - c^2} \cdot x = 2b$  or  $x^3 + 3\sqrt{117} \cdot x = 4$ . And as  $b$  is less than  $c$ , this equation belongs to the two series in the latter case for finding the least root. Hence, the terms of the two series agreeing with the positive and negative terms of the series in Art. 106, they will stand thus :

By

By the 1st series	By the 2d series
A = .3333333	B = .0020406
C = .0000330	D = .0000007
.3333663 - log. 7.5229217	.0020413 - - log. 3.3099068
$\frac{4b}{\sqrt{c^2}} = \sqrt{\frac{512}{121}} - - .02088282$	$\frac{4b}{\sqrt{c^2}} = \sqrt{\frac{512}{121}} - - - .02088282$
series = - .5392000 7.7317499	series = + .0033016 - 3.5187350
X = + .2712508	X = - .2712508
r = - .2679492 the least root	r = - .2679492 the same root.

Agreeing with the same root found in Ex. 4. Art. 106.

131. But the same root has been found to be  $-2 + \sqrt{3}$  in Art. 59, and hence we obtain the sums of these two particular series, thus,

$$\frac{\sqrt[3]{13} - \sqrt[3]{9} + 2 - \sqrt[3]{3}}{8} \sqrt[3]{121} = \frac{1}{3} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 2^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 11^4} + 8\&c.$$

$$\frac{\sqrt[3]{13} - \sqrt[3]{9} - 2 + \sqrt[3]{3}}{8} \sqrt[3]{121} = \frac{2 \cdot 5 \cdot 2^2}{3 \cdot 6 \cdot 9 \cdot 11^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 17 \cdot 2^6}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 21 \cdot 11^4} + 8\&c.$$

132. Also by taking the sum and difference of these two, we have

$$\frac{\sqrt[3]{13} - \sqrt[3]{9}}{4} \sqrt[3]{121} = \frac{1}{3} + \frac{2 \cdot 5 \cdot 2^2}{3 \cdot 6 \cdot 9 \cdot 11^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 2^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 11^4} + 8\&c.$$

$$\frac{2 - \sqrt[3]{3}}{4} \sqrt[3]{121} = \frac{1}{3} - \frac{2 \cdot 5 \cdot 2^2}{3 \cdot 6 \cdot 9 \cdot 11^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 2^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 11^4} - 8\&c.$$

And this last expression agrees with what was found in Art. 108.

133. Ex. 2. Again in the equation  $x^3 - 9x = -10$ , we have  $2b = -10$ , and  $3\sqrt[3]{b^2 + c^2} = 9$ ; consequently  $b = -5$ , and  $c^2 = 3^3 - b^2 = 27 - 25 = 2$ , which being less than  $b^2$  or 25, this equation belongs to the first class of series, or that for the greatest root. Now

$$x = \sqrt[3]{b+c} + \sqrt[3]{b-c} = \sqrt[3]{-5+\sqrt{2}} + \sqrt[3]{-5-\sqrt{2}}$$

$$= -\sqrt[3]{5-\sqrt{2}} - \sqrt[3]{5+\sqrt{2}}$$

$$= -\sqrt[3]{3.58578864} - \sqrt[3]{6.41421356}$$

$$= -1.530600 - 1.858009 = -3.388609 = \text{the}$$

root of the equation  $x^3 - 3\sqrt[3]{b^2 - c^2} \cdot x = 2b$ , or

$$x^3 - 3\sqrt[3]{21} \cdot x = -10. \text{ And the terms of the two series are found as in Art. 110, namely } 1 - \frac{2 \cdot 5 \cdot 8c^4}{3 \cdot 6 \cdot 9 \cdot 12b^4} - 8xc.$$

$$= A - C - E - 8xc. = .9997359, \text{ and } \frac{2c^2}{3 \cdot 6b^3} + 8xc. = B + D + 8xc.$$

$$= .0089009. \text{ Also } 4\sqrt[3]{b} = 4\sqrt[3]{-5} = -4\sqrt[3]{5} = -\sqrt[3]{320}.$$

Then

By the 1st series	By the 2d series
.9997359 - log. $\overline{1.9998854}$	.0089009 - log. $\overline{3.9494339}$
- $\sqrt[3]{320}$ - - - $\underline{0.8350500}$	- $\sqrt[3]{320}$ - - - $\underline{0.8350500}$
series = - 6.838098 - $\underline{0.8349354}$	series = - .060881 - $\underline{2.7844839}$
X = + $\underline{3.388609}$	X = - $\underline{3.388609}$
- 3.449489 the greatest root	- 3.449490 the same root,

And these values of the greatest root are nearly the same with that found in Art. 110.

134. But in Art. 61, the same root was found to be  $-1 - \sqrt{6}$ , hence we obtain the sums of these first two particular series; and by the addition and subtraction of these two arise the other two following them, namely,

$$\frac{1 + \sqrt{6} + \sqrt[3]{5} + \sqrt{2} + \sqrt[3]{5} - \sqrt{2}}{4\sqrt[3]{5}} = 1 - \frac{2 \cdot 5 \cdot 8 \cdot 2^2}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 5^4} - \&c.$$

$$\frac{1 + \sqrt{6} - \sqrt[3]{5} + \sqrt{2} - \sqrt[3]{5} - \sqrt{2}}{4\sqrt[3]{5}} = \frac{2 \cdot 2}{3 \cdot 6 \cdot 5^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 2^3}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 5^6} + \&c.$$

$$\frac{1 + \sqrt{6}}{2\sqrt[3]{5}} = 1 + \frac{2 \cdot 2}{3 \cdot 6 \cdot 5^2} - \frac{2 \cdot 5 \cdot 8 \cdot 2^2}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 5^4} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 2^3}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 5^6} - \&c.$$

$$\frac{\sqrt[3]{5} + \sqrt{2} + \sqrt[3]{5} - \sqrt{2}}{2\sqrt[3]{5}} = 1 - \frac{2 \cdot 2}{3 \cdot 6 \cdot 5^2} - \frac{2 \cdot 5 \cdot 8 \cdot 2^2}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 5^4} - \&c.$$

And the last but one of these equations agrees with one found in Art. 112.

135. Ex. 3. Also in the equation  $x^3 - 12x = 9$ , we have  $2b = 9$ , and  $\sqrt[3]{b^2 + c^2} = 4$ ; consequently  $b = \frac{9}{2}$ , and  $c^2 = 4^3 - b^2 = 64 - \frac{81}{4} = \frac{175}{4}$ , which being greater than  $b^2$  or  $\frac{81}{4}$ , this case belongs to the second class of series, or that of the least roots. Now here  $x = \sqrt[3]{c + b} - \sqrt[3]{c - b} = \sqrt[3]{\frac{\sqrt{175} + 9}{2}} - \sqrt[3]{\frac{\sqrt{175} - 9}{2}} = \sqrt[3]{11 \cdot 11 \cdot 4378} - \sqrt[3]{2 \cdot 11 \cdot 4378} = 2 \cdot 2316619 - 1 \cdot 2834950 = \cdot 9481669 =$  the root of the equation  $x^3 - 3\sqrt[3]{b^2 - c^2} \cdot x = 2b$  or  $x^3 + 3\sqrt[3]{\frac{47}{2}} \cdot x = 9$ . And the terms of the two series being found as in Art. 113, namely,  $A + C + E + \&c. = \cdot 34051$ , and  $B + D + F + \&c. = \cdot 03071$ , also  $\frac{4b}{\sqrt[3]{c^2}}$  being  $= \frac{36}{\sqrt[3]{250}}$ , we shall have

By

By the 1st series	By the latter series
$\cdot 34051 - \log. \overline{1}^{\circ} 3321299$	$\cdot 03071 - \log. \overline{2}^{\circ} 4872798$
$\frac{36}{\sqrt[3]{350}} - - - \cdot 7082798$	$\frac{36}{\sqrt[3]{350}} - - - \cdot 7082798$
series = $- \cdot 1739442 - \cdot 2404097$	series = $+ \cdot 1568771 - \overline{1}^{\circ} 1955596$
X = $+ \cdot 948167$	X = $- \cdot 9481669$
$- \cdot 791274$ the least root	$- \cdot 7912898$ the same root

Which nearly agree with the same root found in Art. 113.

136. But in Art. 62 the same root was found to be  $\frac{3-\sqrt{21}}{2}$ , hence then we shall have these first two following equations, and by means of their sum and difference we obtain the other two :

$$\frac{\sqrt[3]{20\sqrt{7+36}} - \sqrt[3]{20\sqrt{7-36}} + \sqrt[3]{21} - 3}{7^2} \sqrt[3]{350} = \frac{1}{3} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 81^2}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 175^2} \&c.$$

$$\frac{\sqrt[3]{20\sqrt{7+36}} - \sqrt[3]{20\sqrt{7-36}} - \sqrt[3]{21} + 3}{7^2} \sqrt[3]{350} = \frac{2 \cdot 5 \cdot 81}{3 \cdot 6 \cdot 9 \cdot 175} + \&c.$$

$$\frac{\sqrt[3]{20\sqrt{7+36}} - \sqrt[3]{20\sqrt{7-36}}}{36} \sqrt[3]{350} = \frac{1}{3} + \frac{2 \cdot 5 \cdot 81}{3 \cdot 6 \cdot 9 \cdot 175} + \&c.$$

$$\frac{\sqrt[3]{21} - 3}{36} \sqrt[3]{350} = \frac{1}{3} - \frac{2 \cdot 5 \cdot 81}{3 \cdot 6 \cdot 9 \cdot 175} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 81^2}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 175^2} - \&c.$$

And the last of these agrees with one found in Art. 115.

137. Ex. 4. In the equation  $x^3 - 15x = 22$ , we have  $2b = 22$ , and  $\sqrt[3]{b^2 + c^2} = 5$ ; consequently  $b = 11$ , and  $c^2 = 5^3 - b^2 = 125 - 121 = 4$ , which being less than  $b^2$  or 121, this belongs to the first class of series, or that for the greatest root.

Now

Now  $x = \sqrt[3]{b+c} + \sqrt[3]{b-c} = \sqrt[3]{13} + \sqrt[3]{9} = 4.4314186 =$   
the root of the equation  $x^3 - 3\sqrt[3]{117} \cdot x = 22$ . And  
the terms of the two series being found as in Art. 117,  
we have the first  $= A - C - \&c. = 1 - .0000450 =$   
 $.9999550$ , and the second  $= B + D + \&c. = .0036731 +$   
 $.0000008 = .0036739$ . Also  $4\sqrt[3]{b} = 4\sqrt[3]{11} = \sqrt[3]{704}$ .  
Hence,

By the 1st series	By the 2d series
$.9999550 - \log. \bar{1}9999805$	$.0036739 - \log. \bar{3}5651273$
$\sqrt[3]{704} - - - 0.9491909$	$\sqrt[3]{704} - - - 0.9491909$
series $= + 88955200 - 0.9491714$	series $= + .0326827 \quad \bar{2}5143182$
X $= - 4.4314186$	X $= + 4.4314186$
$+ 4.4641014$ greatest root	$+ 4.4641013$ the same root

Which nearly agree with the same root found in  
Art. 117.

138. But in Art. 64, the same root was found to be  
 $1 + \sqrt{12} = 1 + 2\sqrt{3}$ , hence we obtain these two first  
equations following, and their sum and difference give  
the other two :

$$\frac{1 + \sqrt{12} + \sqrt[3]{13} + \sqrt[3]{9}}{4\sqrt[3]{11}} = 1 - \frac{2 \cdot 5 \cdot 8 \cdot 2^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 11^4} - \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 17 \cdot 20 \cdot 2^8}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 21 \cdot 24 \cdot 11^8} - \&c.$$

$$\frac{1 + \sqrt{12} - \sqrt[3]{13} - \sqrt[3]{9}}{4\sqrt[3]{11}} = \frac{2 \cdot 2^2}{3 \cdot 6 \cdot 11^2} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 2^6}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 11^6} + \&c.$$

$$\frac{\sqrt[3]{13} + \sqrt[3]{9}}{2\sqrt[3]{11}} = 1 - \frac{2 \cdot 2^2}{3 \cdot 6 \cdot 11^2} - \frac{2 \cdot 5 \cdot 8 \cdot 2^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 11^4} - \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 2^6}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 11^6} - \&c.$$

$$\frac{1 + \sqrt{12}}{2\sqrt[3]{11}} = 1 + \frac{2 \cdot 2^2}{3 \cdot 6 \cdot 11^2} - \frac{2 \cdot 5 \cdot 8 \cdot 2^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 11^4} + \frac{2 \cdot 5 \cdot 8 \cdot 11 \cdot 14 \cdot 2^6}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15 \cdot 18 \cdot 11^6} - + \&c.$$

The last of which agrees with one found in Art. 119.



And thus we may find the sums of as many series of these kinds as we please ; as well as the sum of any of the general series, by means of the roots of given cubic equations. As to the summation of other forms of series by means of the roots of equations of other orders, I shall perhaps treat of them on some future occasion.

